THE PHYSICAL SOCIETY OF LONDON.

PROCEEDINGS.

VOLUME XIII.—PART I.

OCTOBER 1894.

LONDON:

PRINTED BY TAYLOR AND FRANCIS, RED LION COURT, FLEET STREET.

1894.

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1894–95.

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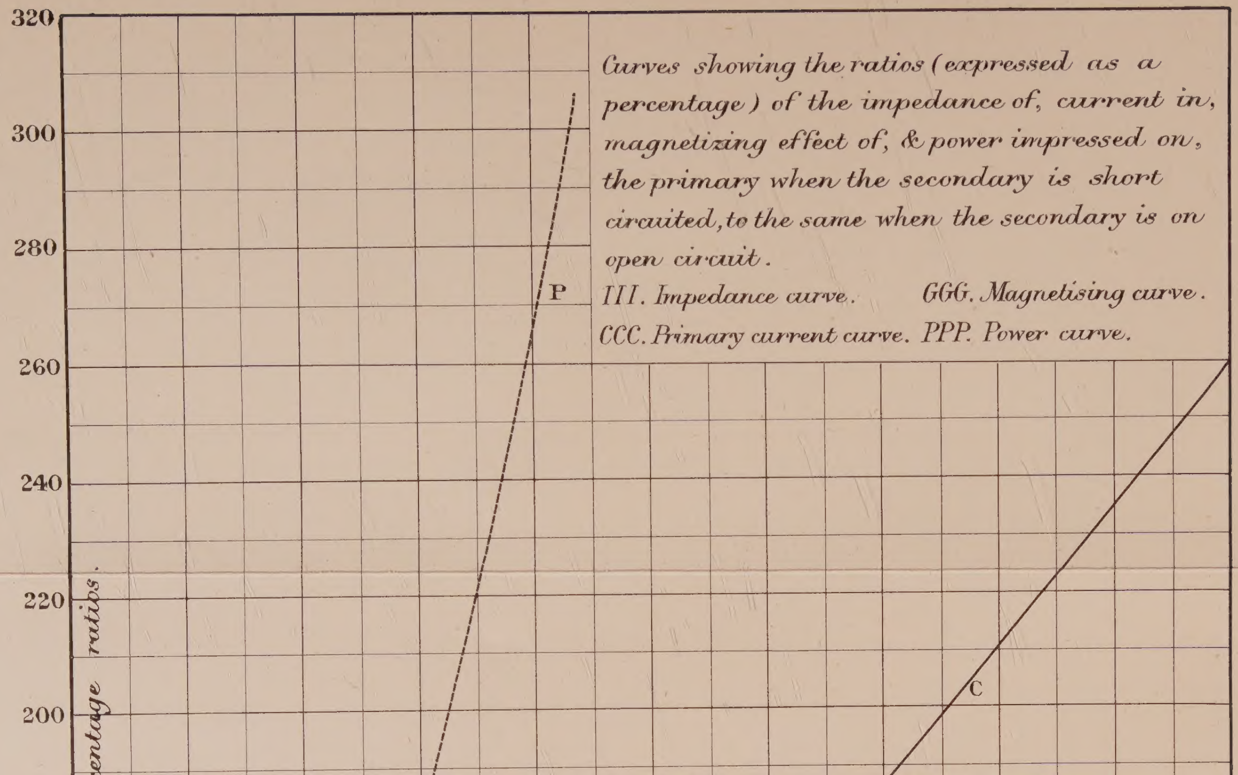
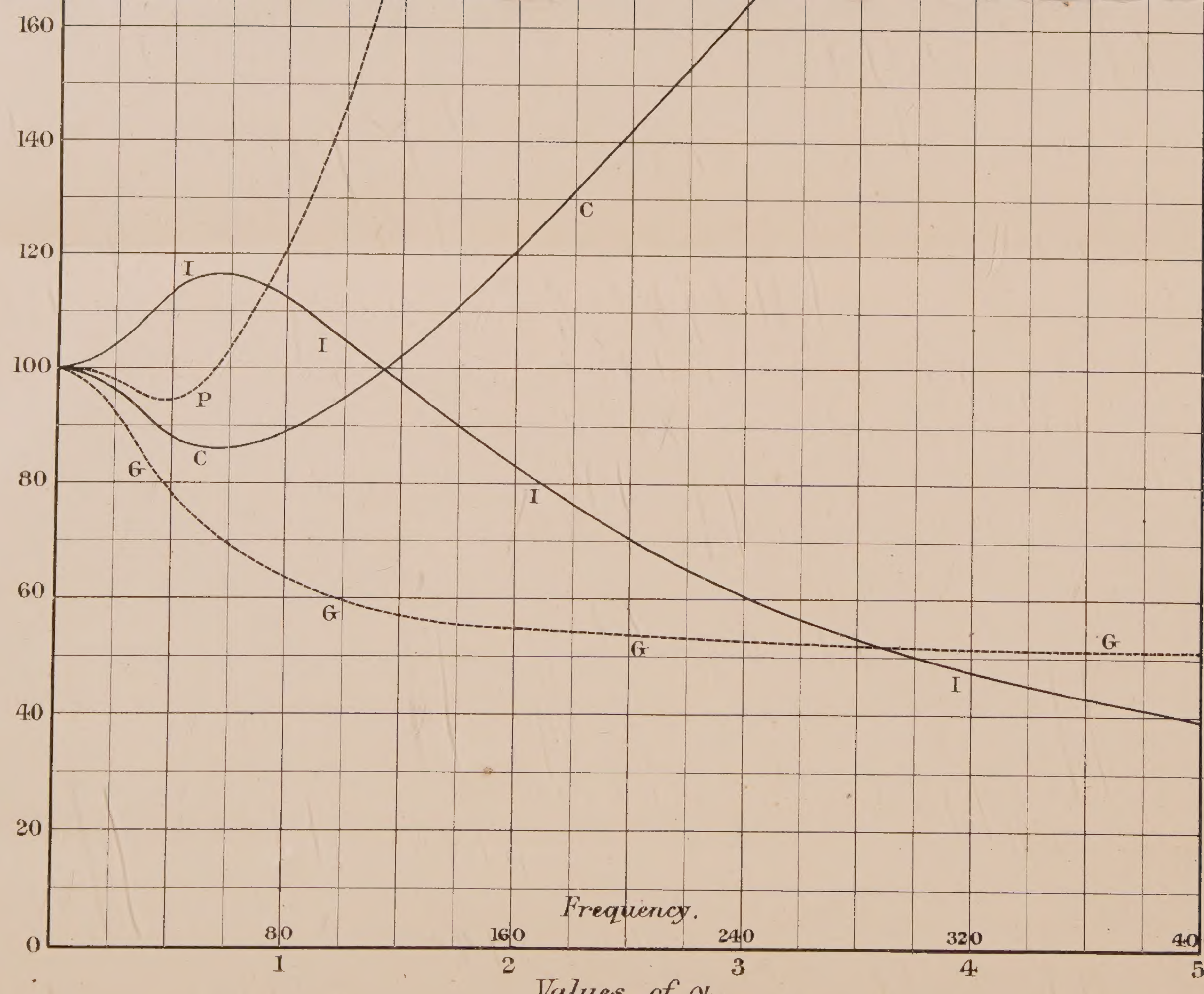


Curves showing the ratios (expressed as a percentage) of the impedance of, current in, magnetizing effect of, & power impressed on, the primary when the secondary is short circuited, to the same when the secondary is on open circuit.

III. Impedance curve. GGG. Magnetising curve.
CCC. Primary current curve. PPP. Power curve.



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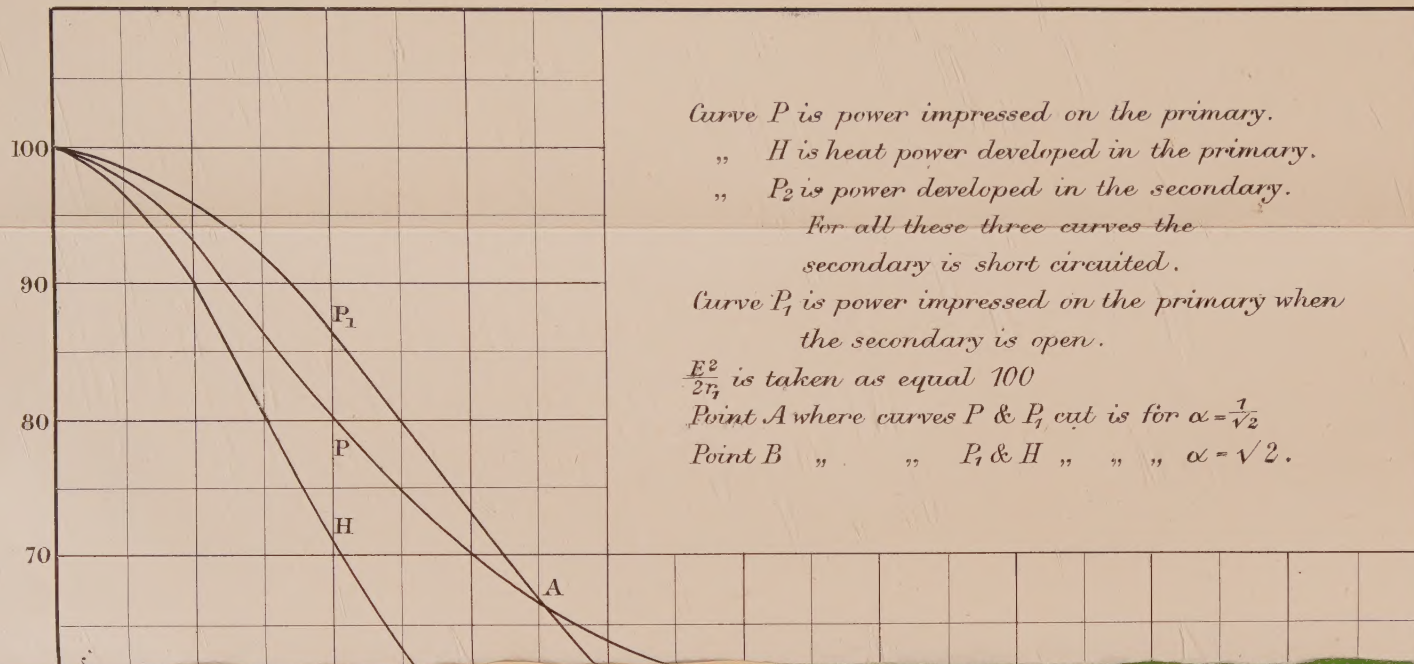
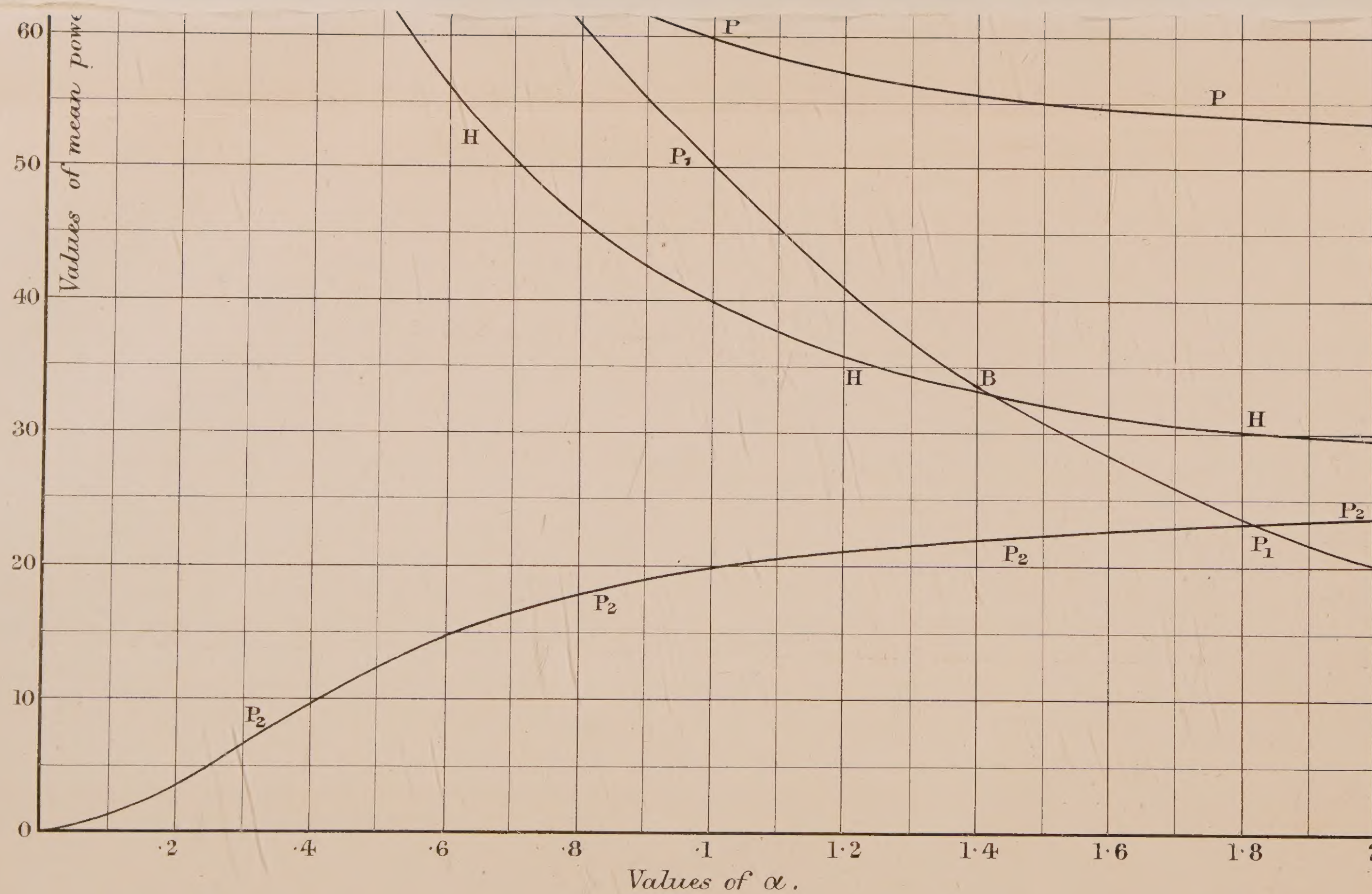

100
90
80
70
P_1
P
H
A
Curve P is power impressed on the primary.
" H is heat power developed in the primary.
" P_2 is power developed in the secondary.
For all these three curves the secondary is short circuited.
Curve P_1 is power impressed on the primary when the secondary is open.
$\frac{E^2}{2r_1}$ is taken as equal 100
Point A where curves P & P_1 cut is for $\alpha = \frac{1}{\sqrt{2}}$
Point B " " P_1 & H " " " $\alpha = \sqrt{2}$.



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PROCEEDINGS

OF

THE PHYSICAL SOCIETY

OF LONDON.

JANUARY 1894.

I. *On the Behaviour of an Air-Core Transformer when the Frequency is below a certain Critical Value.* By E. C. RIMINGTON*.

[Plates I. & II.]

It is usually supposed in the case of a transformer whose primary is connected to terminals having an alternating potential-difference of constant value between them, that the apparent impedance of the primary is diminished on closing the secondary. Under certain conditions, however, this is not the case, as the following investigation will show.

Let r_1 be the resistance of the primary circuit;
L its inductance;
r_2 the resistance of the secondary circuit;
N its inductance;
M the mutual inductance between the two coils.

The coefficients of induction are assumed constant in the following investigation, a result that can only be obtained in practice when coils not containing iron cores are employed. A pure sine-function alternating P.D. is also assumed.

Let $p=2\pi n$, where n is the frequency of alternation.

Let e be the value of the P.D. at any instant t, and E its maximum value.

* Read October 27, 1893.

Let c_1 and c_2 be the currents in the primary and secondary circuits respectively, C_1 and C_2 being their maxima.

Let $I_1 = \sqrt{r_1^2 + p^2L^2}$, the impedance of the primary;
and $I_2 = \sqrt{r_2^2 + p^2N^2}$, the impedance of the secondary.

We have the well-known equations :—

$$L\frac{dc_1}{dt} + M\frac{dc_2}{dt} + c_1r_1 = e; \quad . \quad . \quad . \quad . \quad (1)$$

$$N\frac{dc_2}{dt} + M\frac{dc_1}{dt} + c_2r_2 = 0. \quad . \quad . \quad . \quad . \quad (2)$$

Differentiate (1) with respect to t, and multiply by N; differentiate (2) and multiply by M; then on subtraction we obtain

$$(LN - M^2)\frac{d^2c_1}{dt^2} + Nr_1\frac{dc_1}{dt} - Mr_2\frac{dc_2}{dt} = N\frac{de}{dt}. \quad . \quad (3)$$

Multiply (1) by r_2 and add to (3). This gives

$$(LN - M^2)\frac{d^2c_1}{dt^2} + (Nr_1 + Lr_2)\frac{dc_1}{dt} + r_1r_2c_1 = r_2e + N\frac{de}{dt}. \quad (4)$$

Similarly we obtain

$$(LN - M^2)\frac{d^2c_2}{dt^2} + (Nr_1 + Lr_2)\frac{dc_2}{dt} + r_1r_2c_2 = -M\frac{de}{dt}. \quad (5)$$

Now it is obvious, if the P.D. be a pure sine function and the coefficients constants, that the currents must also be pure sine functions differing only in phase from the P.D.

Assume *, then,

* This assumption will evidently give a particular solution to equation (4), viz. $c_1 = C_1 \sin pt$.

The complete solution is obtained by adding to this the solution of equation (4), assuming the right-hand member zero. So that the complete solution to (4) is

$$c_1 = C_1 \sin pt + Ae^{-\frac{k+h}{2}t} + Be^{-\frac{k-h}{2}t},$$

where
$$k = \frac{Nr_1 + Lr_2}{LN - M^2},$$

and
$$h = \frac{\sqrt{(Nr_1 - Lr_2)^2 + 4r_1r_2M^2}}{LN - M^2}.$$

The constants A and B depend on the phase of the P.D. at the instant the coil is switched on. The exponential terms (since they are both real and negative) rapidly die away, so that practically $c_1 = C_1 \sin pt$ after a short time has elapsed. The same remarks apply to the value of c_2.

$c_1 = C_1 \sin pt, \quad c_2 = C_2 \sin(pt+\theta), \text{ and } e = E\sin(pt+\phi),$

$$\frac{dc_1}{dt} = pC_1 \cos pt \quad \text{and} \quad \frac{d^2c_1}{dt^2} = -p^2C_1 \sin pt\,;$$

also $$\frac{de}{dt} = pE\cos(pt+\phi).$$

Inserting these values in equation (4) gives

$$C_1[\{r_1r_2 - p^2(LN-M^2)\}\sin pt + p(Nr_1+Lr_2)\cos pt]$$
$$= Er_2 \sin(pt+\phi) + EpN\cos(pt+\phi). \quad . \quad . \quad . \quad (6)$$

For shortness, let

a denote $r_1r_2 - p^2(LN-M^2)$, and

b denote $p(Nr_1+Lr_2)$.

Then (6) may be written

$$C_1\sqrt{a^2+b^2}\sin(pt+\psi) = EI_2\sin(pt+\phi+\chi), \quad . \quad (7)$$

where

$$\tan\psi = \frac{b}{a} \quad \text{and} \quad \tan\chi = \frac{pN}{r_2}.$$

As (7) must hold for all values of t, it follows that

$$C_1\sqrt{a^2+b^2} = EI_2,$$

or

$$C_1 = \frac{EI_2}{\sqrt{a^2+b^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and that $\psi = \phi + \chi$, or $\phi = \psi - \chi$.

Hence

$$\tan\phi = \frac{\frac{b}{a} - \frac{pN}{r_2}}{1+\frac{bpN}{ar_2}} = \frac{br_2 - apN}{ar_2 + bpN} = \frac{pL - \frac{p^3NM^2}{I_2^2}}{r_1 + \frac{p^2r_2M^2}{I_2^2}}. \quad . \quad (9)$$

From (9) it is evident that the difference in phase between the primary current and the P.D. is always diminished on closing the secondary, since, when the latter is open

$$\tan\phi = \frac{pL}{r_1}.$$

In the same manner from (5) we obtain

$$C_2\sqrt{a^2+b^2}\sin(pt+\theta+\psi) = -pME\cos(pt+\phi)$$
$$= pME\sin\left(pt+\phi+\frac{3\pi}{2}\right).$$

Hence
$$C_2 = \frac{pME}{\sqrt{a^2+b^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and
$$\theta + \psi = \phi + \frac{3\pi}{2},$$

but
$$\psi = \phi + \chi.$$

Hence
$$\theta = \frac{3\pi}{2} - \chi = \pi + \left(\frac{\pi}{2} - \chi\right),$$

or θ is greater than π and less than $\frac{3\pi}{2}$. Also

$$\tan\theta = \cot\chi = \frac{r_2}{pN}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Now from equation (8),
$$C_1 = \frac{E}{\frac{\sqrt{a^2+b^2}}{I_2}}.$$

Call I the apparent impedance of the primary when the secondary is closed (I_1 is its impedance with secondary open). Then

$$I = \frac{\sqrt{a^2+b^2}}{I_2},$$

or
$$\begin{aligned} I^2 I_2^2 &= a^2 + b^2 \\ &= r_1^2 r_2^2 + p^4(LN - M^2)^2 - 2p^2 r_1 r_2 (LN - M^2) \\ &\quad + p^2 N^2 r_1^2 + p^2 L^2 r_2^2 + 2p^2 r_1 r_2 LN \\ &= I_1^2 I_2^2 - p^2 M^2 \{ p^2(2LN - M^2) - 2r_1 r_2 \}, \end{aligned}$$

or
$$I^2 = I_1^2 - \frac{p^2 M^2}{I_2^2} \{ p^2(2LN - M^2) - 2r_1 r_2 \}. \quad . \quad . \quad (12)$$

Equation (12) shows that I will only be less than I_1 when the quantity inside the brackets is positive; so that, if $2r_1 r_2 > p^2(2LN - M^2)$, I^2 is greater than I_1^2, and hence $I > I_1$, or the impedance of the primary is *increased* on closing the secondary.

For convenience let $\alpha_1 = \frac{pL}{r_1}$, and $\alpha_2 = \frac{pN}{r_2}$. α_1 is of course the tangent of the angle of lag of the primary current behind the P.D. when the secondary is unclosed, while $\frac{\pi}{2} + \tan^{-1}\alpha_2$ is the phase-angle between the primary and secondary currents when the secondary is closed.

Let $M=\beta\sqrt{LN}$, so that β represents the ratio of the magnetic induction passing through the secondary to that through the primary, and is of course less than unity, also $100(1-\beta)$ is the percentage magnetic leakage*. Substituting these values in (12) it becomes

$$\left(\frac{I}{I_1}\right)^2=1+\frac{\beta^2\alpha_1\alpha_2\{2-\alpha_1\alpha_2(2-\beta^2)\}}{(1+\alpha_1^2)(1+\alpha_2^2)} \quad . \quad . \quad . \quad (13)$$

To make $\frac{I}{I_1}$ greater than unity, obviously

$$\alpha_1\alpha_2 \text{ must be less than } \frac{2}{2-\beta^2},$$

* This is only the case when the two coils are equal in dimensions and similar in shape; otherwise the ratio of the total lines of magnetic induction through the secondary to those through the primary, when a current flows in the primary, will not be the same as the ratio of the lines through the primary to those through the secondary when there is a current in the latter.

β is the geometrical mean of these two ratios. Thus: let n_1 be the number of turns in the primary and G_1 some constant depending on its shape and size, then the magnetic induction through the primary $=G_1n_1$, and $L=G_1n_1^2$. The induction through the secondary $=G_2n_2$, and $N=G_2n_2^2$, where the constant G_2 depends on the shape and size of the secondary. Let β_1 be the fraction of the primary induction that threads the secondary, and β_2 the fraction of the secondary induction that threads the primary. Then

$$M=G_1n_1\beta_1n_2=G_2n_2\beta_2n_1.$$

Hence $$M^2=\beta_1\beta_2G_1G_2n_1^2n_2^2,$$

or $$M=\sqrt{\beta_1\beta_2}\sqrt{LN};$$

so that $$\beta=\sqrt{\beta_1\beta_2}.$$

Also $$G_1\beta_1=G_2\beta_2, \text{ or } \frac{\beta_1}{\beta_2}=\frac{G_2}{G_1}.$$

So that for coils of the same shape and size,

$$\beta_1=\beta_2=\beta, \text{ since } G_1=G_2.$$

When the coils are of different shapes or sizes,

$$\frac{\beta_1}{\beta_2}=\frac{G_2}{G_1}=\frac{N}{L}\cdot\left(\frac{n_1}{n_2}\right)^2;$$

also if the number of turns in the primary and secondary is equal,

$$\frac{\beta_1}{\beta_2}=\frac{N}{L}.$$

or the critical value of

$$\alpha_1 = \frac{2}{\alpha_2(2-\beta^2)}; \ (\alpha_2 \text{ given});$$

and that of

$$\alpha_2 = \frac{2}{\alpha_1(2-\beta^2)}; \ (\alpha_1 \text{ given}).$$

When the primary and secondary are identical or have the same shape and coil-volume *, and the secondary when closed is short-circuited,

$$\alpha_1 = \alpha_2 = \alpha.$$

Then

$$\left(\frac{I}{I_1}\right)^2 = 1 + \frac{\beta^2\alpha^2\{2-\alpha^2(2-\beta^2)\}}{(1+\alpha^2)^2}, \quad . \quad . \quad . \quad (14)$$

and the critical value of $\alpha = \sqrt{\frac{2}{2-\beta^2}}$.

When $\beta = 1$, or there is no magnetic leakage, the critical value of $\alpha = \sqrt{2}$.

To find the value of α_1 *that will make* $\frac{I}{I_1}$ *a maximum,* α_2 *and* β *being given.*

Obviously from (13) $\frac{I}{I_1}$ is a maximum when $\left(\frac{I}{I_1}\right)^2$ is a maximum, that is, when $\frac{\alpha_1\{2-\alpha_1\alpha_2(2-\beta^2)\}}{1+\alpha_1^2}$ is a maximum, since α_2 and β are constants.

Differentiating with respect to α_1 and equating to zero gives

$$\alpha_1 = \frac{\sqrt{4+\alpha_2^2(2-\beta^2)^2} - \alpha_2(2-\beta^2)}{2}. \quad . \quad . \quad (15)$$

If $\beta = 1$, or there be no magnetic leakage,

$$\alpha_1 = \frac{\sqrt{4+\alpha_2^2} - \alpha_2}{2} \quad \text{or} \quad \frac{\alpha_1}{1-\alpha_1^2} = \frac{1}{\alpha_2}. \quad . \quad (15\,a)$$

If the coils have the same time-constants, or $\alpha_1 = \alpha_2 = \alpha$,

$$\alpha = \frac{1}{\sqrt{3-\beta^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15\,b)$$

* Relative difference of thickness of insulation being supposed negligible, if different-sized windings are used.

and
$$\frac{I}{I_1}=\sqrt{1+\frac{\beta^2}{4-\beta^2}}=\frac{2}{\sqrt{4-\beta^2}}$$
if also $\beta=1$,
$$\alpha=\frac{1}{\sqrt{2}},\text{ and }\frac{I}{I_1}=\sqrt{1\cdot\dot{3}}=1\cdot155.$$

Obviously, if α_1 and β are known the value of α_2 that makes $\frac{I}{I_1}$ a maximum is by symmetry,
$$\alpha_2=\frac{\sqrt{4+\alpha_1^2(2-\beta^2)^2}-\alpha_1(2-\beta^2)}{2}\quad\text{or}\quad\frac{1-\alpha_2^2}{\alpha_2(2-\beta^2)}=\alpha_1,$$
or in the case of no magnetic leakage, $\frac{\alpha_2}{1-\alpha_2^2}=\frac{1}{\alpha_1}$.

If α_1 and α_2 are both variables, we have the two equations to be satisfied, viz.:—
$$\alpha_2=\frac{1-\alpha_1^2}{\alpha_1(2-\beta^2)}\quad\text{and}\quad\alpha_1=\frac{1-\alpha_2^2}{\alpha_2(2-\beta^2)},$$
$$\therefore\quad\alpha_1\alpha_2(2-\beta^2)=1-\alpha_1^2=1-\alpha_2^2,$$
$$\therefore\quad\alpha_1=\alpha_2=\alpha\text{ say},$$
and
$$\alpha=\frac{1-\alpha^2}{\alpha(2-\beta^2)}\quad\text{or}\quad\alpha=\frac{1}{\sqrt{3-\beta^2}}.$$

So that to get $\frac{I}{I_1}$ a maximum, the primary and secondary should have the same value of α, each equal to $\frac{1}{\sqrt{3-\beta^2}}$, or in the case of no magnetic leakage $=\frac{1}{\sqrt{2}}$; in which case $\frac{I}{I_1}$ will be 1·155, or a $15\frac{1}{2}$ per cent. increase in impedance caused by short-circuiting the secondary, and this is the greatest that can be obtained.

Consider, then, the case of a transformer having coils with equal time-constants, and suppose there is no magnetic leakage.

For values of α below $\sqrt{2}$ the impedance is increased, and putting $\beta=1$ in equation (14) gives
$$\frac{I}{I_1}=\frac{\sqrt{1+4\alpha^2}}{1+\alpha^2}.$$

From this equation curve I I I, Plate I., has been plotted; the abscissæ are values of α, and the ordinates values of $\frac{\mathrm{I}}{\mathrm{I}_1}$, expressed as percentages. For a transformer in which $\frac{\mathrm{L}}{r_1}=\frac{\mathrm{N}}{r_2}=\frac{1}{500}$ about, a second set of figures giving the frequency is also marked.

The curve C C C, Plate I., is the ratio of C the primary current when the secondary is closed to C_1 the same secondary open, and is evidently plotted from the equation

$$\frac{\mathrm{C}}{\mathrm{C}_1}=\frac{\mathrm{I}_1}{\mathrm{I}}=\frac{1+\alpha^2}{\sqrt{1+4\alpha^2}}.$$

Power Distribution in above Transformer.

Let P be the power impressed on the primary when the secondary is closed ;

P_1 the same when the secondary is open ;

H the heat-power developed in the primary when the secondary is closed ;

P_2 the power developed in the secondary.

Then

$$\mathrm{P}=\frac{p}{2\pi}\cdot\frac{\mathrm{E}^2\mathrm{I}_2}{\sqrt{a^2+b^2}}\int_0^{\frac{2\pi}{p}}\sin pt\,.\sin(pt+\phi)\,.\,dt$$

$$=\frac{\mathrm{E}^2(r_1\mathrm{I}_2{}^2+r_2p^2\mathrm{M}^2)}{2(a^2+b^2)}$$

$$=\frac{\mathrm{E}^2}{2r_1}\cdot\frac{1+2\alpha^2}{1+4\alpha^2}. \qquad (16)$$

$$\mathrm{H}=\frac{p}{2\pi}\cdot\frac{\mathrm{E}^2\mathrm{I}_2{}^2r_1}{a^2+b^2}\int^{\frac{2\pi}{p}}\sin^2 pt\,.\,dt$$

$$=\frac{\mathrm{E}^2\mathrm{I}_2{}^2r_1}{2(a^2+b^2)}=\frac{\mathrm{E}^2}{2r_1}\cdot\frac{1+\alpha^2}{1+4\alpha^2}. \qquad (17)$$

$$\mathrm{P}_2=\frac{p}{2\pi}\cdot\frac{(p\mathrm{ME})^2r_2}{a^2+b^2}\int_0^{\frac{2\pi}{p}}\sin^2(pt+\theta)\,dt$$

$$=\frac{p^2\mathrm{M}^2\mathrm{E}^2r_2}{2(a^2+b^2)}=\frac{\mathrm{E}^2}{2r_1}\cdot\frac{\alpha^2}{1+4\alpha^2}. \qquad (18)$$

Obviously $P = P_2 + H$, unless there are masses of metal present, in which eddy currents are developed, or the frequency is so great that appreciable energy is radiated.

$$P_1 = \frac{p}{2\pi} \cdot \frac{E^2}{I_1} \int_0^{\frac{2\pi}{p}} \sin pt \,.\, \sin(pt + \phi)\, dt$$

$$= \frac{E^2 r_1}{2I_1^2} = \frac{E^2}{2r_1} \cdot \frac{1}{1+\alpha^2}. \qquad \qquad (19)$$

Hence, from (16) and (19),

$$\frac{P}{P_1} = \frac{(1+2\alpha^2)(1+\alpha^2)}{1+4\alpha^2} = \frac{1+3\alpha^2+2\alpha^4}{1+4\alpha^2}, \qquad (20)$$

and from this equation (20) the curve marked PPP (Plate I.) is plotted.

The curves for P, H, P_2, and P_1 are plotted in Plate II. for values of α up to 2.

Magnetizing Effect of the Coils.

Let g be the number of effective current-turns at any instant when the secondary is closed, and G its maximum value.

Then $$g = n_1 c_1 + n_2 c_2,$$

where n_1 and n_2 are the number of turns in the primary and secondary respectively, and

$$G = \sqrt{n_1^2 C_1^2 + n_2^2 C_2^2 + 2n_1 n_2 C_1 C_2 \cos\theta},$$

where θ is the phase-angle between the two currents.

If we assume the primary and secondary to occupy equal volumes, and we can neglect the relative difference in thickness of the insulation of the two coils, $\dfrac{n_1}{n_2} = \sqrt{\dfrac{r_1}{r_2}}$,

$$\therefore\ G = n_1 \sqrt{C_1^2 + C_2^2 \frac{r_2}{r_1} + 2C_1 C_2 \sqrt{\frac{r_2}{r_1}} \,.\, \cos\theta},$$

$$= \frac{n_1 E}{\sqrt{a^2+b^2}} \sqrt{I_2^2 + p^2 M^2 \frac{r_2}{r_1} - 2p^2 MN \sqrt{\frac{r_2}{r_1}}};$$

or (since M^2 is assumed $=$LN)

$$G=\frac{n_1 E r_2}{\sqrt{a^2+b^2}}=\frac{n_1 E}{r_1\sqrt{1+4\alpha^2}}.$$

Call G_1 the maximum value of g when the secondary is open; then

$$G_1=\frac{n_1 E}{I_1}=\frac{n_1 E}{r_1\sqrt{1+\alpha^2}},$$

$$\therefore \quad \frac{G}{G_1}=\sqrt{\frac{1+\alpha^2}{1+4\alpha^2}}. \quad \ldots \ldots \quad (21)$$

Evidently, then, the magnetizing effect is always diminished on closing the secondary.

The curve GGG (Plate I.) is plotted from equation (21).

It will be seen, on inspecting Plate I., that in the cases of the impedance-curve and the current-curve the critical value for $\alpha=\sqrt{2}=1{\cdot}414$, while the value of α that makes them a maximum and a minimum respectively is $\alpha=\frac{1}{\sqrt{2}}={\cdot}707$. This latter value of α is the critical value for the curve PPP, while, as may be seen by differentiating equation (20) and equating to zero, the value of α that makes this curve a minimum is $\frac{\sqrt{\sqrt{3}-1}}{2}$, or ${\cdot}4278$.

Consider, now, the case of primary and secondary having the same time-constants but with magnetic leakage.

Then we know the *critical value of* $\alpha=\sqrt{\frac{2}{2-\beta^2}}$.

The value of α to make $\frac{I}{I_1}$ a maximum from (15b)$=\frac{1}{\sqrt{3-\beta^2}}$.

Let γ represent the ratio of the former value of α to the latter, then

$$\gamma=\sqrt{\frac{6-2\beta^2}{2-\beta^2}};$$

also the maximum value of $\frac{I}{I_1}=\frac{2}{\sqrt{4-\beta^2}}$.

From equation (11),

$$\tan\theta=\frac{r_2}{pN}=\frac{1}{\alpha};$$

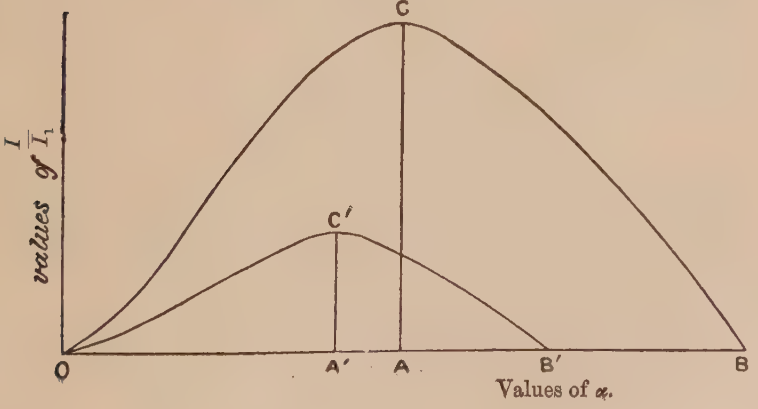
and θ is the angle by which the secondary current is in

advance of the primary, and it lies between π and $\frac{3\pi}{2}$; it therefore follows that the angle by which the secondary current lags behind the primary lies between $\frac{\pi}{2}$ and π, and is $\frac{\pi}{2} + \tan^{-1}\alpha$. The following Table gives values of the above quantities for values of β from 1 to ·1.

β.	Critical values of α.	Values of α for max. $\frac{I}{I_1}$.	γ.	Max. values of $\frac{I}{I_1}$.	Critical phase-angle.	Phase-angle for max. $\frac{I}{I_1}$.
1·0	$1·414 = \sqrt{2}$	$·707 = \frac{1}{\sqrt{2}}$	2	1·155	144° 44′	125° 16′
·9	1·300	·676	1·920	1·120	142 26	124 4
·8	1·210	·650	1·865	1·090	140 26	123 2
·7	1·150	·630	1·825	1·070	139 0	122 13
·6	1·105	·615	1·795	1·050	137 51	121 36
·5	1·070	·605	1·770	1·032	136 56	121 11
·4	1·045	·595	1·760	1·020	136 16	120 45
·3	1·023	·586	1·750	1·010	135 39	120 22
·2	1·010	·581	1·740	1·007	135 17	120 10
·1	1·002	·578	1·730	1·001	135 4	120 2

It will be seen from the column of values for γ that when $\beta = 1$, or there is no magnetic leakage, the critical value of α is twice its value for maximum $\frac{I}{I_1}$; and the effect of leakage is to diminish this number, so that when $\beta = ·1$ it is reduced to 1·73. This is shown in the subjoined figure.

Fig. 1.



The curve OCB represents the critical part of the curve III (Plate I.) when there is no leakage, and A comes midway between O and B, so that $OA = \frac{1}{2} OB$; also $OB = \sqrt{2} = 1{\cdot}414$, $OA = {\cdot}707$, and, if the point O represent 100 divisions, $AC = 15\frac{1}{2}$ divisions.

The curve $OC'B'$ represents the critical part when $\beta = {\cdot}1$; $OB' = 1{\cdot}002$ and $OA' = {\cdot}578 = {\cdot}577\, OB'$; also $A'C' = {\cdot}12$ division. Hence we see that the effect of magnetic leakage is to shift the point A' corresponding to the maximum values of $\frac{I}{I_1}$ from midway between O and B' and towards B'.

The author was enabled, through the kindness of Dr. Fleming, to try an experiment in the meter-testing room of the Electric Supply Company, using a Kelvin balance for measuring the currents through the primary coil. The alternating P.D. was obtained from the terminals of a transformer capable of giving over 100^{A}; and as the maximum current ever taken was $6\frac{1}{2}^{A}$ about, the P.D. may be assumed constant. The frequency was 83·3.

The air-core transformer used for the experiment consisted of two coils wound one inside the other, and of No. 20 B.W.G. cotton-covered wire. The inner coil was used as primary and the outer as secondary.

Each coil consisted of 5 layers, of 125 turns per layer.

Calculating the time-constants or values of $\frac{L}{R}$ for the coils by Perry's approximate formula, they worked out as ·00121 for the inner coil or primary, and ·00152 for the outer coil or secondary.

$$p = 2\pi \times 83{\cdot}3 = 523.$$

Hence

$$\alpha_1 = {\cdot}00121 \times 523 = {\cdot}633,$$

$$\alpha_2 = {\cdot}00152 \times 523 = {\cdot}795.$$

The values of α_1 and α_2 were probably smaller than these values, as the coils became fairly warm from working; moreover the primary had the leads and the resistance of a Kelvin balance in series with it.

Take, then, α_1 as ·5, and α_2 as ·7.

The observed value of $\frac{I}{I_1}$ was 1·032.

Substituting these values in equation (13) makes $\beta = \cdot 57$, or a magnetic leakage of 43 per cent., if the P.D. were a true sine function. This seems a rather large value for the leakage; and it is probable that the leakage was considerably less than this, but that the P.D. was not a pure sine function, and on this account the ratio $\frac{I}{I_1}$ was less than it otherwise would have been.

The above experiment was only a rough one, but it showed an increase of 3·2 per cent. in the impedance of the primary on closing the secondary: moreover the time-constants of the coils were not suited for giving the best effect with the frequency employed; but, as is seen from the previous theoretical investigation, by employing a primary and a secondary having equal time-constants suitably related to the frequency, and a pure sine function P.D., an increase of impedance of from 10 to 12 per cent. ought to be obtained; 15½ per cent. increase could never be obtained in practice, as there must always be some magnetic leakage.

In transformers with iron cores this effect would be likely to escape notice, as the values of $\frac{L}{R}$ would be so large that the critical frequency would be very small, so that for all frequencies employed in practice the impedance of the primary would diminish on closing the secondary. The iron core would also distort the current from a pure sine function.

The above investigations may be with advantage treated by the geometrical or clock-face diagram method, as it shows at once to the eye the relations between the different quantities and the phase-angles.

Consider first the case of an identical primary and secondary and no magnetic leakage; or $L = N = M$, and $r_1 = r_2$.

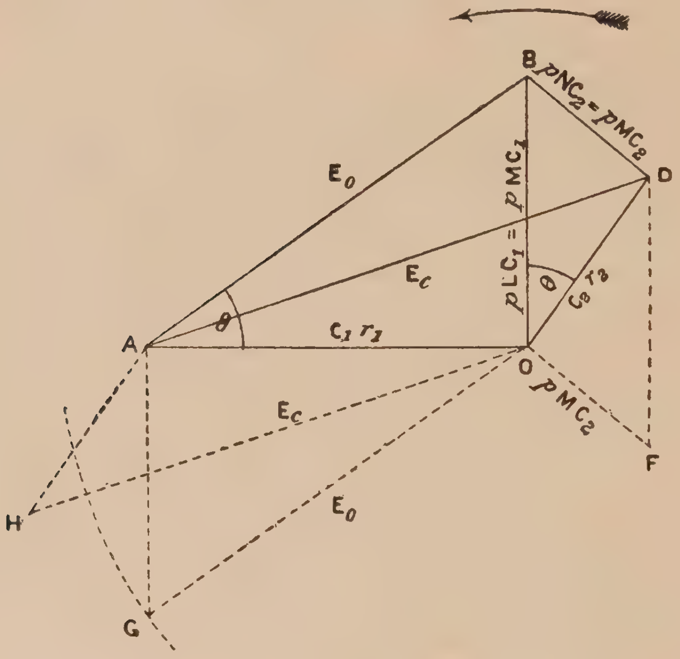
As before, call $\frac{pL}{r_1} = \frac{pN}{r_2} = \alpha$, and let $\tan^{-1}\alpha = \theta$, or

$$\tan\theta = \alpha = \frac{pL}{r_1} = \frac{pN}{r_2}.$$

Draw a line $OA = C_1 r_1$. Draw OB at right angles to OA and 90° behind OA as regards the direction of rotation, shown by the arrow; make $OB = pLC_1$. Then AB represents

the value of the P.D. at the terminals of the primary when the secondary is open or of E_0, and its actual position on the diagram is the dotted line OG parallel to AB.

Fig. 2.



Now OB also represents pMC_1, since M=L. Draw OD making an angle θ (equal to the angle OAB) with OB and behind it; drop the perpendicular BD on to OD. Then OD represents C_2r_2, and $BD = pNC_2 = pMC_2$. Draw OF 90° behind OD and equal to $pMC_2 = BD$. Then OF represents the back E.M.F. in the primary due to the mutual induction, the E.M.F. in the same due to self-induction being $OB = pLC_1$, and their resultant is obviously OD.

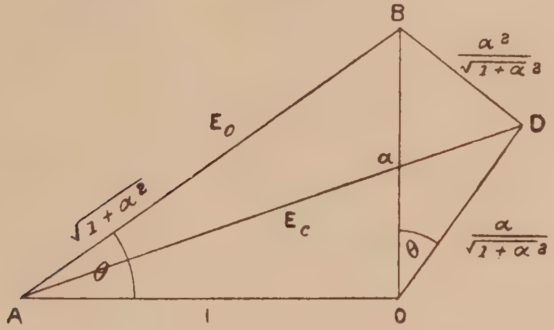
Join AD. Then AD must represent in magnitude and OH (equal and parallel to AD) in direction in the diagram the terminal P.D. when the secondary is closed or E_c, because C_1r_1 or OA is the resultant of OH, the impressed P.D., and OD the resultant E.M.F. in the primary due to self and mutual inductions.

Now, as the diagram is drawn, E_c is greater than E_0, the primary current being constant, so that if the terminal P.D. were kept constant the primary current with the secondary closed divided by the same with the secondary open would equal $\frac{E_0}{E_c}$, or the impedance ratio $\frac{1}{I_1} = \frac{E_c}{E_0}$.

The actual lines that need be drawn are shown in full, the dotted ones being superfluous except for explanation, and they will in future be omitted.

If we take $C_1 r_1$ equal to unity we shall have the diagram fig. 3.

Fig. 3.



Now $AD^2 = AO^2 + OD^2 - 2AO \cdot OD \cos AOD$

$$= 1 + \frac{\alpha^2}{1+\alpha^2} - \frac{2\alpha}{\sqrt{1+\alpha^2}} \cos\left(\frac{\pi}{2} + \theta\right)$$

$$= 1 + \frac{\alpha^2}{1+\alpha^2} + \frac{2\alpha}{\sqrt{1+\alpha^2}} \sin\theta.$$

Now $\tan\theta = \alpha$, hence $\sin\theta = \frac{\alpha}{\sqrt{1+\alpha^2}}$;

$$\therefore\ AD^2 = 1 + \frac{3\alpha^2}{1+\alpha^2} = \frac{1+4\alpha^2}{1+\alpha^2}, \text{ or } AD = \sqrt{\frac{1+4\alpha}{1+\alpha^2}}.$$

Now AD represents E_c and AB represents E_0. So that

$E_c >=< E_0$ accordingly as $\sqrt{\frac{1+4\alpha^2}{1+\alpha^2}} >=< \sqrt{1+\alpha^2}$,

or as

$$1 + 4\alpha^2 >=< (1+\alpha^2)^2,$$
$$1 >=< \pm(1-\alpha^2).$$

Taking the minus sign,

$$E_c >=< E_0 \text{ as } \alpha <=> \sqrt{2}.$$

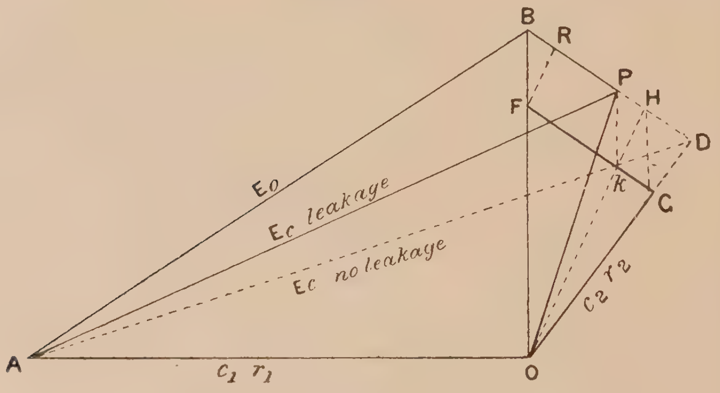
Taking the plus sign,

$$E_c >=< E_0 \text{ as } \alpha >=< 0;$$

that is $E_c > E_0$ if α lies between $\sqrt{2}$ and 0, not including these values.

Case of an identical Primary and Secondary with Magnetic Leakage.

Fig. 4.



As before make $OA = C_1 r_1$, $OB = pLC_1$, then $AB = E^0$. Let $M = \beta L = \beta N$.

In OB take a point F such that $OF = \beta \times OB = pMC_1$. Draw OD making angle $\theta =$ angle OAB with OB and drop perpendicular BD. Drop perpendicular FG on OD, then $FG = pNC_2$, and $OG = C_2 r_2$.

Through G draw GH parallel to OB and cutting BD in H. Join OH and let it cut FG in K, through K draw KP parallel to OB and cutting BD in P.

Through F draw FR parallel to OH and meeting BD in R.

Then
$$\frac{BH}{RH} = \frac{OB}{OF} = \frac{1}{\beta}.$$

Therefore $RH = \beta . BH$, but $BH = FG$, so

$$RH = \beta . FG = \beta p NC_2 = pMC_2.$$

Again, RH = FK = BP since FR is parallel to OH and KP to OB.

Now OP is the resultant of $OB = pLC_1$, and $BP = pMC_2$. Join AP.

Then AP represents E_c.

If there were no leakage AD would represent E_c. Now AP is obviously less than AD, or the effect of leakage is to diminish the ratio $\frac{E}{E_0}$, that is the ratio of $\frac{I}{I_1}$.

Obviously from the figure the greater the leakage the lower F, the smaller FG and BH, the smaller also the ratio $\frac{BP}{BH}$, and consequently the nearer P to B, that is the nearer the value of E_c to that of E_0.

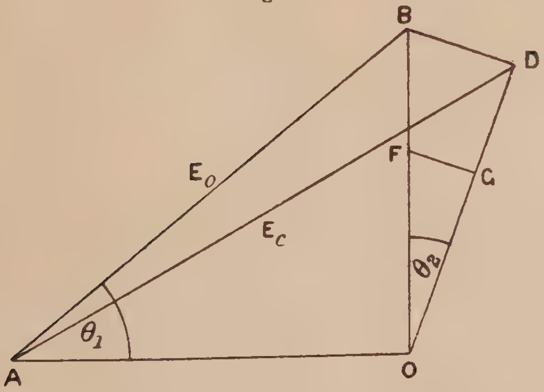
Case of Two Coils with different Time-Constants without Magnetic Leakage.

$$\text{Primary} \quad L, r_1, \text{ and } \alpha_1 = \frac{pL}{r_1}; \ \tan\theta_1 = \alpha_1.$$

$$\text{Secondary} \ N, r_2, \text{ and } \alpha_2 = \frac{pN}{r_2}; \ \tan\theta_2 = \alpha_2.$$

$$M^2 = LN, \text{ so that } \frac{M}{N} = \frac{L}{M}.$$

Fig. 5.



$$OA = C_1 r_1, \quad OB = pLC_1, \quad AB = E_0.$$

$$\text{Make } \frac{OF}{OB} = \frac{M}{L}. \quad \therefore \ OF = pMC_1.$$

Draw OD making angle $\theta_2 = \tan^{-1}\alpha_2$ with OB, and drop perpendiculars FG and BD on to OD.

Then $\quad$ $FG = pNC_2$ and $OG = C_2 r_2$.

Now

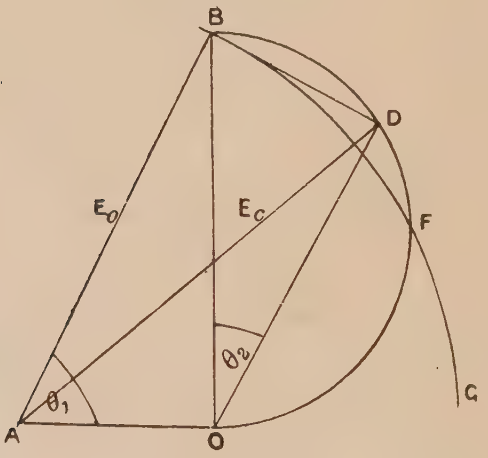
$$\frac{M}{N} = \frac{L}{M} = \frac{OB}{OF} = \frac{BD}{FG} \text{ (by similar triangles).}$$

$$\therefore\ BD = FG \times \frac{M}{N} = pNC_2 \times \frac{M}{N} = pMC_2.$$

Hence OD represents the back E.M.F. in primary due to self and mutual inductions, and $AD = E_c$.

However large the value of α_1 (unless it were infinite, which is of course not possible) it is always possible by making α_2 sufficiently small to obtain E_c greater than E_0. In fig. 6 the

Fig. 6.

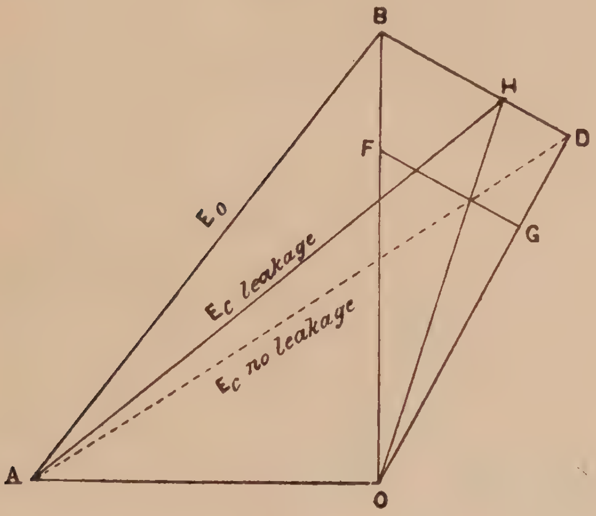


angle $OAB = \theta_1 = \tan^{-1}\alpha_1$ and $AB = E_0$. On OB describe the semicircle BDO. With centre A and radius AB describe the the circle BFG cutting BDO in F. Now by taking any point D in the circle BDO between B and F and joining OD and AD, the latter, which represents E_c, will be greater than AB or E_0, and the tangent of the angle BOD represents α_2.

Obviously when D coincides with F, $E_c = E_0$, or closing the secondary makes no difference to the impedance of the primary; while if D be between F and O, E_c is less than E_0

Case of Two Coils with different Time-Constants with Magnetic Leakage.

Fig. 7.



Let $M^2=\beta^2 LN$ so that $\frac{M}{N}=\beta^2\frac{L}{M}$.

Construction is the same as in the last case, or $\frac{OF}{OB}=\frac{M}{L}$;

H is taken so that $\frac{BH}{BD}=\beta^2$.

Now $\frac{BD}{FG}=\frac{L}{M}$; $\therefore\ \frac{BH}{FG}=\beta^2\frac{L}{M}=\frac{M}{N}$,

so that

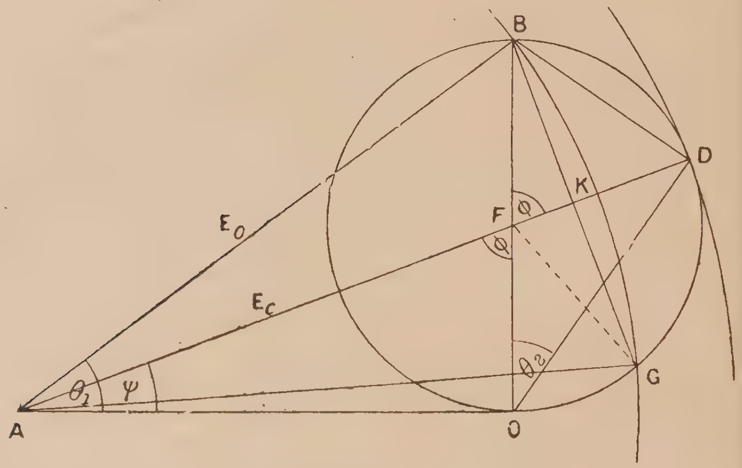
$$BH=\frac{M}{N}\cdot FG=\frac{M}{N}\cdot pNC_2=pMC_2.$$

$\therefore$ OH is the back E.M.F in the primary due to self and mutual inductions, and consequently $AH=E_c$.

$AD=E_c$ if there be no leakage.

PROBLEM I. *A transformer with a given primary (or α_1 known), to find the value of α_2 for secondary so as to make $\frac{E_c}{E_0}$ or $\frac{I}{I_1}$ a maximum. Also to find the critical value of α_2.*
No magnetic leakage.

Fig. 8.



Let $\theta_1 = \tan^{-1} \alpha_1$ which is known.

Draw $OA = C_1 r_1$; draw OB at right angles to OA and equal to pLC_1 or $\alpha_1 C_1 r_1$. Then $AB = E_0$ and the angle OAB $= \theta_1$.

On OB as diameter describe the circle ODB with centre F; then $OF = \frac{1}{2}OB$.

Join AF and produce it to cut the circle in D.

Then AD is the greatest straight line that can be drawn from A to cut the circle ODB, since if two circles touch the line through their centres passes through the point of contact

Join BD and OD, then the angle BOD $= \theta_2$ and $\frac{BD}{OD} = \tan \theta_2 =$ the required value of α_2.

Let the angle BFD $= \phi$ and the angle OAD $= \psi$.

Then

$$\theta_2 = \tfrac{1}{2}\phi = \tfrac{1}{2}\left(\frac{\pi}{2} - \psi\right) \text{ or } 2\theta_2 = \frac{\pi}{2} - \psi.$$

Now $$\frac{\tan\psi}{\tan\theta}=\frac{OF}{OB}=\tfrac{1}{2},\ \text{or}\ \cot\psi=\frac{2}{\tan\theta_1};$$

also

$$\tan 2\theta_2=\tan\left(\frac{\pi}{2}-\psi\right)=\cot\psi=\frac{2}{\tan\theta_1}.$$

Hence $$\frac{2\tan\theta_2}{1-\tan^2\theta_2}=\frac{2}{\tan\theta_1},$$

or $$\frac{\alpha_2}{1-\alpha_2^2}=\frac{1}{\alpha_1},$$

or $$\alpha_2=\frac{\sqrt{4+\alpha_1^2}-\alpha_1}{2}$$

Obviously from the figure, however small θ_1 may be, θ_2 is never greater than 45°, or α_2 greater than unity.

To find the Maximum Value of $\frac{E_c}{E_0}$ *or* $\frac{I}{I_1}$.

Call $OA=1$, then $OB=\alpha_1$, and $AB=\sqrt{1+\alpha_1^2}$.

Now

$$AD=AF+FD=\sqrt{1+\frac{\alpha_1^2}{4}}+\frac{\alpha_1}{2}=\frac{\sqrt{4+\alpha_1^2}+\alpha_1}{2}.$$

But

$$\frac{E_c}{E_0}=\frac{AD}{AB}=\frac{\sqrt{4+\alpha_1^2}+\alpha_1}{2\sqrt{1+\alpha_1^2}},$$

or the maximum value of $\frac{I}{I_1}=\frac{\sqrt{4+\alpha_1^2}+\alpha_1}{2\sqrt{1+\alpha_1^2}}$.

Hence

$$\left(\frac{I}{I_1}\right)^2_{\text{max.}}=\frac{4+\alpha_1^2+\alpha_1^2+2\alpha_1\sqrt{4+\alpha_1^2}}{4(1+\alpha_1^2)}$$

$$=1+\frac{\alpha_1(\sqrt{4+\alpha_1^2}-\alpha_1)}{2(1+\alpha_1^2)}.$$

Now since $\sqrt{4+\alpha_1^2}$ is always greater than α_1, whatever value α_1 has (except ∞) $\left(\frac{I}{I_1}\right)^2_{\text{max.}}$ is always greater than unity, or $\left(\frac{I}{I_1}\right)_{\text{max.}}$ is always greater than unity.

To find the Critical Value for α_2.

With centre A and radius AB describe a circle cutting the circle BDO in G. Then the angle GOB is the critical value of θ_2, and $\tan\theta_2 = \frac{GB}{OG} =$ the critical value of α_2.

Join AG and FG. Now AG=AB, FG=FB, and AF is common.

∴ the angle AFG=the angle AFB, and hence the angle GFK=the angle BFK.

But BF=FG, and FK is common.

∴ BK=KG, but BF=OF.

Hence OG is parallel to FK.

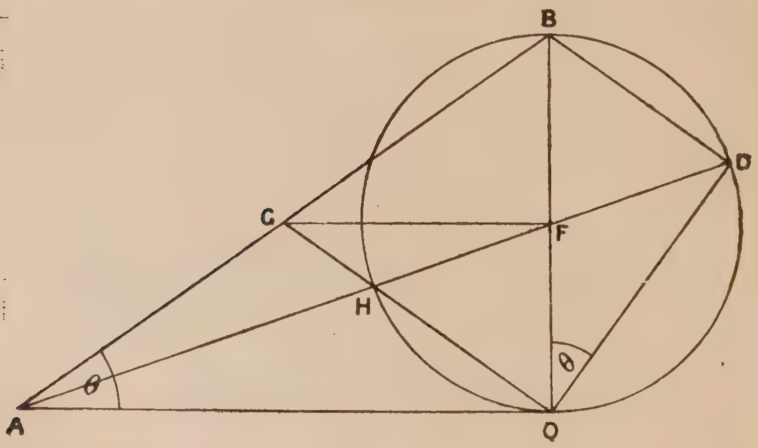
∴ the critical value of θ_2 or the angle GOB=the angle BFD=ϕ.

But $\phi=2\theta_2$ for maximum $\frac{I}{I_1}$.

∴ the critical value of θ_2=twice the value of θ_2 that gives $\frac{I}{I_1}$ a maximum.

PROBLEM II. *A transformer with primary and secondary having the same time-constants, to find the value of α to give maximum $\frac{I}{I_1}$. No magnetic leakage.*

Fig. 9.



Construction. Draw a circle with centre F, and take any diameter OB, draw OA at right angles to OB.

With centre F and radius 3OF describe a circle cutting OA in A.

Join AB and AF and produce AF to meet the circle in D. Join BD and OD.

Then the angle OAB = the angle BOD = θ, and the required value of α

$$=\tan\theta=\frac{OB}{OA}=\frac{BD}{OD}$$

Proof. Call the angle BOD = θ.

Join OH and produce it to meet AB in G: join FG.

Now AH = HD by construction, and OH is parallel to BD.

Hence GH is parallel to BD, and therefore AG = GB.

Again, since AG = GB, and OF = FB, GF is parallel to OA.

$\therefore$ the angle BFG = the angle BOA = a right angle.

Now OF = FB and FG is common: hence

$$OG = GB = AG.$$

$\therefore$ the angle $GAO = GOA = \frac{\pi}{2} - GOB = BOD = \theta$, since the angle HOD being in a semicircle $= \frac{\pi}{2}$.

$\therefore$ the angle $BAO = BOD = \theta = \tan^{-1}\alpha$, or the coils have the same time-constant.

Now from the solution of Problem I. the construction is such as to give a maximum value for $\frac{I}{I_1}$, but it is so arranged that while giving a maximum $\frac{I}{I_1}$ the time-constant of the secondary is the same as that of the primary. Q.E.F.

To find α *or* $\tan\theta$.

$$AF = 3OF \text{ (by construction).}$$

Now $AO^2 = AF^2 - OF^2 = 8OF^2$.

$$\therefore AO = 2OF\sqrt{2}.$$

Now $OB = 2OF$ and $\alpha = \tan\theta = \frac{OB}{OA} = \frac{2OF}{2OF\sqrt{2}} = \frac{1}{\sqrt{2}}$.

To find the maximum value of $\frac{I}{I_1}$, *that is of* $\frac{E_c}{E_0}$.

$$AO^2 = 8OF^2 \text{ and } AD = 4OF.$$

Now

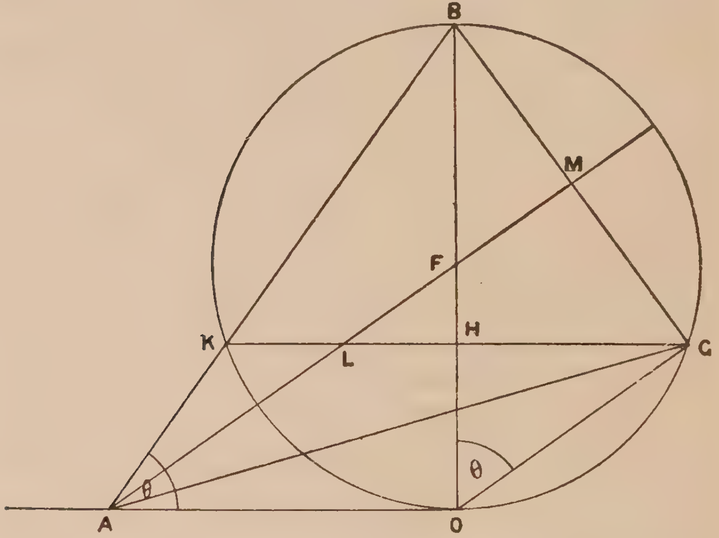
$$AB = \sqrt{AO^2 + OB^2} = \sqrt{8OF^2 + 4OF^2} = \sqrt{12OF^2} = 2OF\sqrt{3}.$$

Also

$$\frac{I}{I_1} = \frac{E_c}{E_0} = \frac{AD}{AB} = \frac{4OF}{2OF\sqrt{3}} = \frac{2}{\sqrt{3}} = 1{\cdot}155.$$

PROBLEM III. *A transformer with primary and secondary having the same time-constants, to find the critical value of* α. *No magnetic leakage.*

Fig. 10.



Construction. Draw any circle with centre F and a diameter OB, draw OA at right angles to OB. In OB take a point H such that BH=2OH or $OH = \frac{1}{3}OB$.

Through H draw KHG parallel to OA and meeting the circle in K and G.

Join BK and produce it to meet OA in A. Join OG and BG.

Then the angle OAB = the angle BOG = θ = the critical angle required, and the critical value of $\alpha = \tan\theta = \frac{OB}{OA}$.

Proof. Firstly, to prove the angles OAB and BOG to be equal.

Since KG is parallel to OA, the angles KHB and GHB are right angles, and KH = HG, also BH is common.

Therefore the angle KBH = GBH.

But AOB is a right angle by construction, and BGO because it is in a semicircle.

Hence the angle BAO = BOG = θ.

Secondly, to prove that AG = AB or $E_c = E_0$, in which case from Problem I. it is known that θ is the critical angle. Join AF and produce it to meet BG in M.

Since KG is parallel to AO by construction,

$$\frac{FH}{FO} = \frac{LH}{AO};$$

but $\quad FH = \frac{1}{3}FO$ (since $OH = \frac{1}{3}OB = \frac{2}{3}FO$);

$$\therefore \quad LH = \tfrac{1}{3}AO.$$

Again, $\dfrac{AO}{KH} = \dfrac{OB}{HB} = \frac{3}{2}$; hence $AO = \frac{3}{2}KH$;

$$\therefore \quad LH = \tfrac{1}{2}KH = \tfrac{1}{2}HG \quad \text{or} \quad LG = 3LH.$$

But $\quad AO = 3LH; \quad \therefore \quad AO = LG;$

or AL is parallel to OG; and the angle AMB = AMG = a right angle.

Also BM = MG and AM is common.

$$\therefore \quad AB = AG \quad \text{or} \quad E_c = E_0.$$

To find the Critical Value of α or tan θ.

Since AF is parallel to OG the angle BFM = θ;

$$\therefore \quad \text{the angle } LFH = \theta = \text{angle } BKH.$$

Hence $$\frac{LH}{FH} = \frac{BH}{KH}$$

Now $\quad KH = 2LH;$

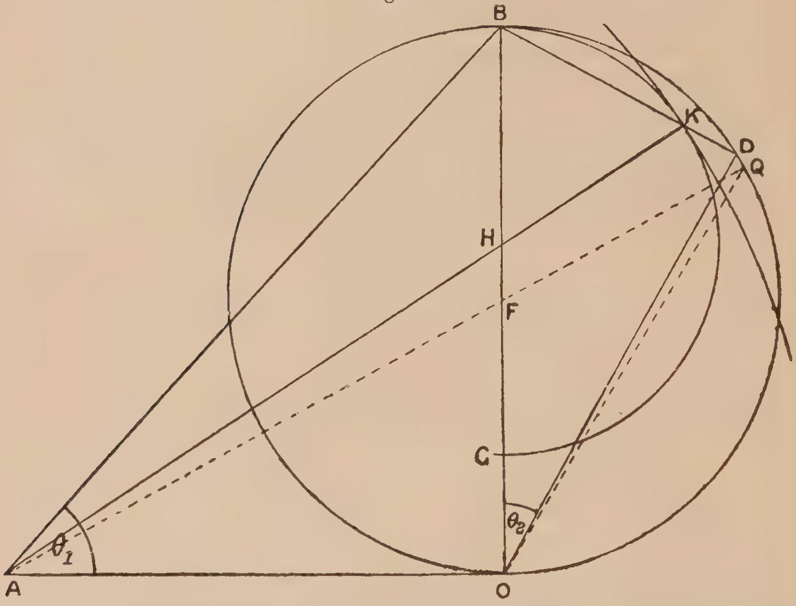
$$\therefore \quad 2LH^2 = FH \,.\, BH = \frac{OF}{3} \cdot \frac{4OF}{3} = \frac{4OF^2}{9},$$

or $$LH = \frac{OF\sqrt{2}}{3}.$$

Now $$\alpha = \tan\theta = \frac{LH}{FH} = \frac{OF\sqrt{2}}{3FH} = \frac{OF\sqrt{2}}{3 \,.\, \frac{OF}{3}} = \sqrt{2}.$$

PROBLEM IV. *A primary with given time-constant, or α_1 known, magnetic leakage or β known, to find the value of α_2 so as to make $\frac{I}{I_1}$ a maximum.*

Fig. 11.



Construction. Draw a circle BDO with diameter OB and centre F; draw OA at right angles to OB and make OA of such length that $\frac{OB}{OA}=\alpha_1$.

Join AB. Then the angle OAB represents θ_1.

From B mark off BH such that

$$\frac{BH}{BF}=\beta^2 \quad (M=\beta\sqrt{LN}).$$

With centre H and radius HB describe circle BKG.

Join AH and produce it to cut circle BKG in K.

Join BK and produce it to cut circle BDO in D.

Join OD, then the angle BOD is the required value of θ_2, and tan BOD$=\frac{BD}{OD}$ the required value of α_2.

Also $\frac{AK}{AB}$ is the maximum value of $\frac{E}{E_0}$ or of $\frac{I}{I_1}$.

Proof. Since $BH=\beta^2 BF$, $BG=\beta^2 BO$, and $BK=\beta^2 BD$ (for join KG, then KG is parallel to DO since the angles at

K and D are both right angles, being in semicircles, and therefore by similar triangles $\frac{BK}{BD}=\frac{BG}{BO}$).

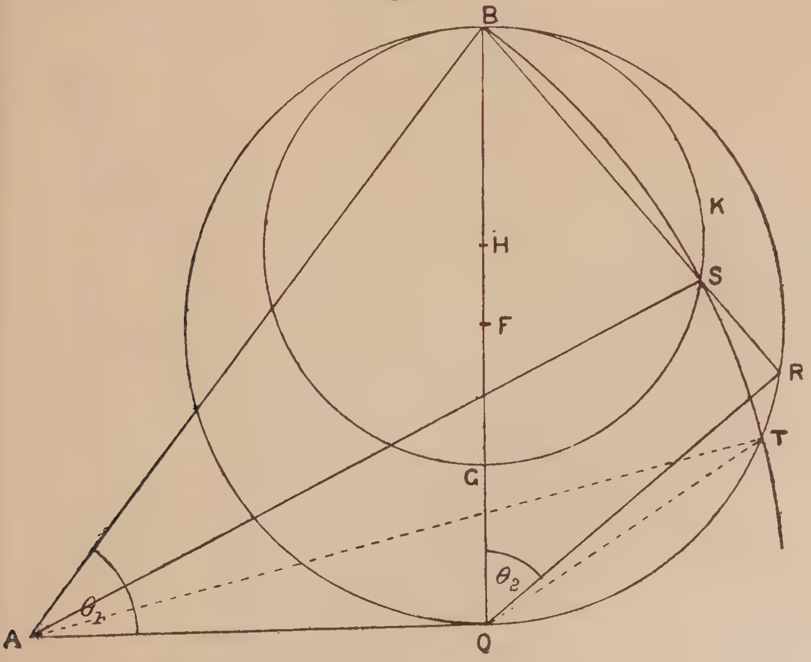
It follows then that, whatever the value of θ_1 the point K lies on the circle BKG when BK represents the back E.M.F. in the primary due to mutual induction, OK the resultant back E.M.F. due to mutual and self-induction, and $AK=E_c$ the impressed potential difference.

With centre A describe a circle touching the circle BKG, then K is the point of contact, since AK passes through H the centre of circle BKG. Therefore AK is the maximum value of E_c, and the angle BOD the value of θ_2, which gives this maximum value for E_c. If there were no leakage the dotted line AQ would represent the maximum value of E_c, and the angle BOQ the corresponding value of θ_2.

It is easily seen from the figure that the effect of leakage is to diminish the maximum value of $\frac{E_c}{E_0}$ or of $\frac{I}{I_1}$, and to diminish also the value of θ_2, and consequently that of α_2, necessary to obtain it.

PROBLEM V. *Same conditions as Problem* IV.; *to obtain the critical value of* α_2.

Fig. 12.



The first part of the construction is the same as in Problem IV.

With centre A and radius AB describe circle BST cutting circles BKG and BTO in S and T respectively. Join BS.

Produce BS to meet circle BTO in R and join OR.

Then the angle BOR is the critical value of θ_2, and that of α_2

$$=\tan\theta_2=\frac{BR}{OR}.$$

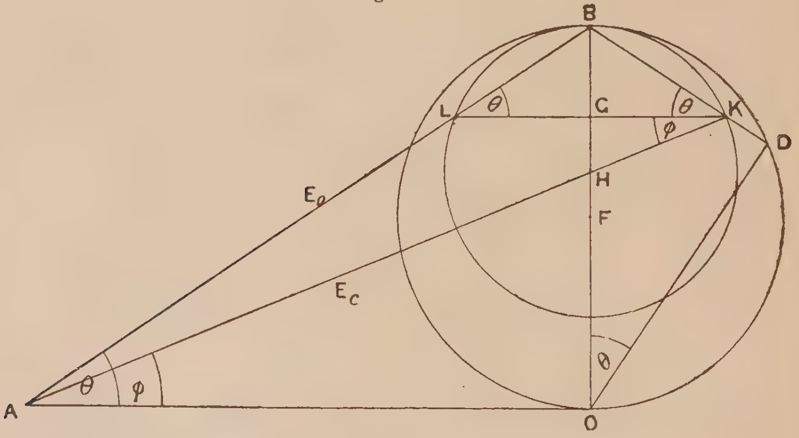
If there were no leakage the angle BOT would be the critical value of θ_2.

The proof is obvious from that of Problem IV.

Here, again, the effect of magnetic leakage is to diminish the critical value of θ_2, and consequently that of α_2.

Problem VI. *A transformer with primary and secondary having the same time-constants, to find the value of a to obtain $\frac{I}{I_1}$ a maximum, the magnetic leakage or value of β being known.*

Fig. 13.



This is the same as Problem II., only there is magnetic leakage.

Construction. Draw a circle BDO with centre F and diameter BO.

Draw OA at right angles to OB.

In OB take a point H such that $\frac{BH}{BF}=\beta^2$.

Take a point G in BH such that $\frac{HG}{GB}=\frac{OH}{OB}$.

(For method of obtaining G geometrically see Problem VII.)

With centre H and radius HB describe circle LBK.

Through G draw LGK parallel to OA and meeting the circle LBK in L and K.

Join BL and produce it to meet OA in A.

Join BK and produce it to meet circle BDO in D. Join OD and AK.

Then $\frac{AK}{AB}$ is the maximum value of $\frac{I}{I_1}$; and the angle

$$BAO=BOD=\theta \text{ is the required angle;}$$

and

$$\tan\theta=\frac{OB}{OA} \text{ is the required value of } \alpha.$$

Proof. Obviously the angle LBG = the angle KBG.

$\therefore$ the angle BAO = the angle BOD = θ.

Call H the point where AK cuts OB.

Now since LK is parallel to OA, the angle BAO = the angle BLK = BKL = θ; and the angle KAO = LKA = ϕ say.

Now

$$\frac{\tan\phi}{\tan\theta}=\frac{OH}{OB}=\frac{HG}{GB}.$$

But by construction, when $\frac{OH}{OB}=\frac{HG}{GB}$, H is the centre of the circle LBK.

$\therefore$ AK is the longest line that can be drawn from A to the circle LBK.

$$\therefore \quad \frac{AK}{AB}=\frac{E_c}{E_0}=\frac{I}{I_1} \text{ is a maximum.}$$

Problem VII.—*Same conditions as Problem VI.; to obtain the critical value of α.*

Construction. Draw with centre F a circle BDO and take a diameter BO, draw OA at right angles.

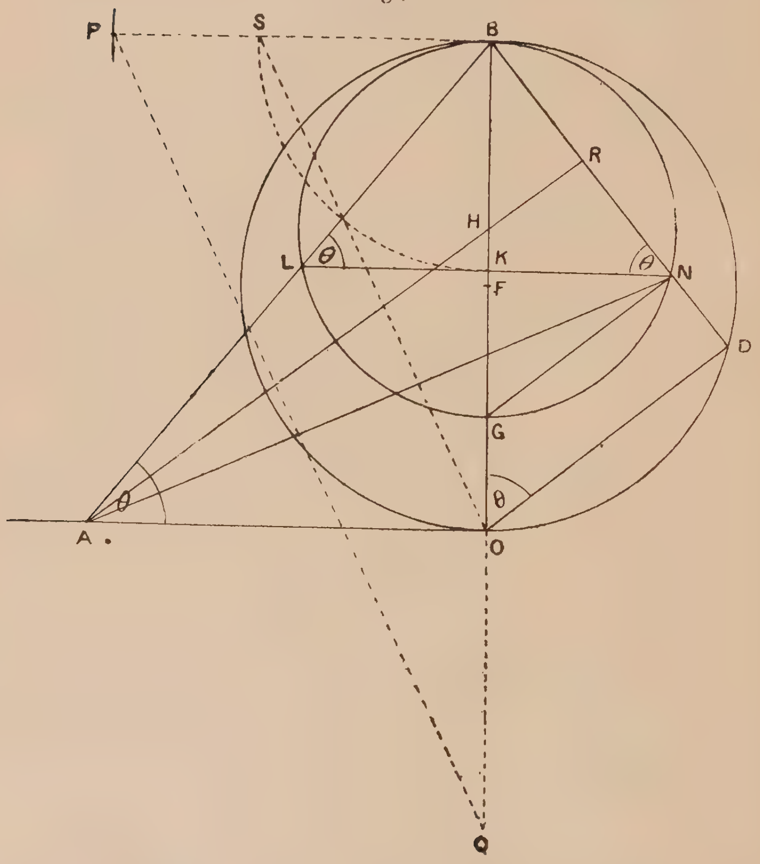
In BO take H such that $\frac{BH}{BF}=\beta^2$.

With H as centre and radius HB describe circle BNGL.

In GB take a point K such that $\frac{GK}{KB}=\frac{OH}{OB}$.

(To obtain K, produce BO to Q making $OQ=OH$, draw

Fig. 14.



$BP=BG$, join PQ, and through O draw OS parallel to PQ, then $BK=BS$.)

Through K draw LKN parallel to OA and meeting circle BNGL in L and N.

Join BL and produce to meet OA in A.

Join BN and produce to meet circle BDO in D. Join OD and AN.

Then the angle $BAO=BOD=\theta$, and

$$AN=AB \quad \text{or} \quad E_e=E_0 \quad \text{or} \quad I=I_1.$$

Proof. Obviously the angle $BAO = BOD$ as before. Join AH and produce it to meet BD in R. Join GN. Now

$$\frac{GK}{KB} = \frac{\tan GNK}{\tan KNB};$$

but

$$KNB = BLK = BAO = \theta.$$

Again,

$$\frac{OH}{OB} = \frac{\tan OAH}{\tan BAO};$$

but

$$\frac{GK}{KB} = \frac{OH}{OB} \text{ by construction.}$$

$$\therefore \quad \frac{\tan OAH}{\tan \theta} = \frac{\tan GNK}{\tan \theta};$$

∴ the angle $GNK =$ the angle OAH, or AR is parallel to GN;

∴ the angles at R are right angles and $BR = RN$.

Also AR is common.

∴ $AB = AN$ or $E_e = E_0$.

In all the above geometrical constructions the phase-angle between the secondary and primary currents is represented by $\frac{\pi}{2} + \theta_2$, or, when the primary and secondary have the same time-constant, by $\frac{\pi}{2} + \theta$.

Discussion.

Prof. Minchin showed that the impedances might be represented by two hyperbolas, having p as abscissæ and the squares of the impedance as ordinates. These could be readily constructed from the data given. A line representing the primary inductance, drawn on the same diagram, intersects one hyperbola, showing that the impedance has always a maximum value. By a simple construction the phase-angle between the primary and secondary currents could be determined for any given conditions.

Dr. Sumpner observed that increased impedance on closing the secondary necessarily meant a decrease in the lag of the primary current behind the primary P.D.

Mr. Blakesley was pleased to see the geometrical method of such service, and thought it much simpler than the analytical one. The reason why increased impedance on closing the secondary of ordinary transformers had not been noticed, was because their lag-angles were very large. In a figure published some years ago to represent the actions of transformers, the angles he had chosen were such as would make the primary impedance increase on closing the secondary. Giving an expression connecting the primary currents on open and closed secondary respectively, he now showed that to get increased impedance the sum of the lag-angles in primary and secondary must exceed 90°. To get large power in the secondary, the primary lag should be nearly 90° and the secondary about 45°. He also pointed out that some of the figures in the paper might be simplified considerably.

Prof. Perry said he had long had the impression that if a sufficiently small current were taken from the secondary, increased impedance would be observable in all cases, and he quoted some numbers he had given in the Phil. Mag. for 1891, showing a decided increase.

Mr. Rimington, in reply, said he was not aware that the effect he had now brought forward had been observed previously. The result was completely worked out analytically before using geometrical methods.

II. *A New Mode of making Magic Mirrors.* *By* J. W. Kearton*.

The first explanation that occurred to me on seeing the Japanese mirror about fourteen months ago was that the face might bear directly invisible differences in polish, which a powerful beam of light would probably convert into visible ones by reflexion on to the screen. To produce such minute differences, it was my intention to take pairs of different metals closely agreeing in colour and reflective power, as silver and platinum, and to deposit electrically in the form of some easily recognizable figure a thin coating of the one

* Read January 26, 1894.

metal on a groundwork of the other. Very faint mercuric staining of bright metallic surfaces was also contemplated. These ideas, however, resolved themselves into a test much simpler, yet involving the same principle. A piece of metal was so polished that in subdued daylight a cross, more finely burnished than the general surface of the plate, could just be distinguished. Reflected on to the screen, the figure came out exceedingly faint; and this fact, apart from the consideration that figures so produced could have only a precarious existence, was sufficient to condemn the hypothesis in question.

This result has a bearing upon another hypothesis—one worthy of prompt burial with the quiddities and essences of the purely deductive method, viz., that the magic-mirror phenomena are due to local molecular rearrangements in the reflecting surface, brought about by unequal cooling of the mass of the mirror. Now, since it is held that the regularity or convexity of the surface is not thereby affected, this molecular rearrangement can put itself in evidence optically only by reflecting more or less light than the parts of the surface unaffected by irregular cooling. But it has been shown that with figures so pronounced as to be directly visible, the electric beam is powerless to produce results comparable in intensity with those given by the Japanese mirror; much less will directly invisible figures of the type under reference come up to the required standard.

The plate used in the foregoing experiment, whilst furnishing no clue in virtue of its polished figure, yet presented evidence that pointed clearly in the direction where a solution of the problem was to be found. Strikingly well-defined lines and dapplings of light were thrown on the screen by light reflected from the disk. These marks were found to correspond to concave strains produced by hammering the plate into rough convexity before scouring down with charcoal. Depressions produced by electro-deposition of silver on a silvered plate, protected in parts by varnish according to the figure desired, were therefore tried; but they invariably deepened into the underlying brass before the sharp edges of the figure were polished away.

Next, a fairly thick coating of silver was deposited on the

plate, and a pointed bit of agate was repeatedly drawn with pressure over certain parts of the porous layer of metal. The figures thus obtained were startling; for the depressions appeared on the screen as reticulated lines of deep shade, which had their analogue in the broken spinal divisions left by the agate point. The bad working of metals electrically deposited caused me to give up the deposition method.

My plate of brass was next slung up in a weak solution of copper sulphate and sulphuric acid on the positive wire of a pint bichromate-cell. After an immersion of four minutes, the parts of the plate not protected by the naphthaline solution of sealing-wax came out beautifully fretted. After several attempts on these lines, and two days after the lecture delivered here by Prof. S. P. Thompson on the 27th of January last on the Magic Mirror, my first success was achieved in the shape of a mirror 2 inches in diameter, representing a stem with leaves and a guide-post standing in a mound of earth. From that mirror to my present one, the first of the new class being completed in the following July, was, however, a far cry. Applied to larger mirrors and figures, the electrical method proved essentially bad. Exactly contrary to what was required, the figures were deeper at the edges of the lines than at the central parts: indeed, a broad line scoured down into two lines defining the lateral limits of the original one.

Nitric acid was finally adopted as the figure-eating agent, care being taken to use good brass and to polish well before immersion, so that the action on the plate might be uniform. Smooth figures were thus produced after immersions of upwards of six seconds in strong acid. The walls of the figures are perpendicular to the surface of the disk, and the determination of their proper height relatively to the breadth of the lines of the figure involved much further labour. The process of scouring with Sheffield lime and swans'-down calico has a double effect—it rounds off the upper rectangular edges of the walls, and at the same time converts the flat floors of the figure into concave depressions, the walls and floors finally merging into one concave sweep. This curve, with narrow lines, is sharper than with broad lines; consequently the latter may disappear, while the former remain

visible. Very brief immersions in the acid are therefore not suitable for figures having lines differing much in breadth. Long immersions, on the other hand, are objectionable both on the ground of the excessive labour in rounding off the edges, and of the well-nigh impossible task of reducing uniformly the deep runnels of the figure by scouring the general surface with charcoal.

Gradations in depth according to the breadth of the several limbs were therefore tried, and produced in the following way:—The figure, say of a tree, is made by removing paraffin wax from the brass plate with a pointed stick of boxwood, and is then fixed by a very short immersion in the acid. The slender branches are now painted over with hot wax, and the broader branches and trunk are reduced in breadth symmetrically by the same means, the plate then being immersed a second time. By a repetition of alternate painting and immersing, all the members of the figure are made to increase in depth by fine gradations from the boundary lines to the central parts. This method, specially applicable to figures with very broad and narrow lines, is, however, rather troublesome, and does not give satisfactory results unless the gradations are minute, which otherwise will be brought out by reflexion on to the screen.

Reverting to single immersions, final experiments were made with plates bearing figures prepared in the usual way, viz. by removal of wax, a record being kept of the strength of the acid and the time of immersion in each case. The result is that for figures with lires ranging from $\frac{1}{16}$ to $\frac{3}{8}$ of an inch in breadth, an immersion of 3 seconds in a solution of 5 volumes of concentrated nitric acid to 2 of water gives a satisfactory depth.

The next step was to produce figures in low relief, which come out on the screen in shade. As there is in this case no intersection of the reflected rays, and, consequently, no blotting out or dimming of the figures, very narrow lines may be used. These are conveniently drawn with a camel's-hair brush and sealing-wax dissolved in naphtha. From 1 to 2 seconds' immersion, according to the breadth of the characters, will be found sufficient.

D 2

Discussion.

Prof. S. P. Thompson said the chief interest of Mr. Kearton's work was that he had succeeded in producing mirrors by a process which Prof. Ayrton had found unsuccessful. The spherical polisher used by Mr. Kearton might have something to do with the result obtained. Some of the mirrors had been gilt after polishing and the reflected pattern improved thereby.

Prof. Ayrton said he was greatly interested to see that mirrors could be produced by the chemical method. The polisher used by the Japanese was the flat end of a tight bundle of special straw cut crosswise. When the true explanation of the magic properties was found out, the chemical method was not pursued further.

The Rev. F. J. Smith mentioned that he had produced magic properties on silvered glass by the inductoscript method. Although no markings could be seen directly, the pattern showed itself when light was reflected from the surface to a screen.

III. *Some Observations on Diffraction.*

By W. B. Croft, *M.A.**

[Plates III.-VI.]

It is proposed to illustrate various forms of this phenomenon by photographs † produced directly from the wave-interference.

After the inauguration of the idea about 1665 by Grimaldi, Hooke, and Huygens, there was little progress, either in extended observation or in philosophical grasp of the principles, until the beginning of this century. Since that time the subject has been treated in two ways.

1st. *The Diffraction of* Fraunhofer *and* Schwerd.

This kind is familiar to many through the observations of Sir John Herschel of Diffraction in a Telescope. It is sometimes described as the Diffraction from Parallel Light.

* Read January 26, 1894.

† It is not convenient to reproduce all the photographs: the selected figures 2, 3, 4, 10 12 13, 71 72, 75, 83 will be found on Plates III.-VI.

Fig. 2.

Fig. 3

Fig. 4.

Fig. 10.

Fig. 12.

Fig. 13.

Fig. 71.

Fig. 72.

Fig. 75.

Fig. 83.

Although more often named after Fraunhofer, as is due to the first great worker in this direction, high tribute is always paid to the more perfect consummation of theory and experiment which was published by Schwerd of Spires in 1835.

A telescope is focussed to a star or a distant point of light, and various limiting apertures are placed on the outer side of the object-glass. Consider, first, the case of two or three narrow parallel lines of light: they constitute an elementary grating, and light from the star passing from them to the object-glass will come as if from several stars lying on either side of the real star, all being spectral images except the central one. The object-glass focusses these at the usual place, then the eyepiece magnifies them and transmits parallel rays of light; so there is a broken line of light across the field, consisting of the plain image at the centre and the spectra of the successive orders on either side. More complex limiting apertures give these radial groups of images of the source of light, of which the number and direction are regulated by the general form of the aperture: thus a triangle will give three lines across the field or a six-rayed star, a square will give a four-rayed star; a circle concentric with the object-glass, however, will give concentric circular spectra.

The objects whose effects have been photographed consist of combinations of thin circular lines of light on a dark glass plate $\frac{3}{4}$ inch diameter; this fits into a cap at the end of a telescope, and each figure in turn is brought to a small opening in front of the centre of the object-glass: this is known as Bridge's apparatus, constructed some years back for this form of diffraction. Fig. 8 almost represents these figures; it has slight traces of diffraction of the other kind. Figs. 1–7 (figs. 2, 3, 4 on Pl. III.) represent the Fraunhofer diffraction of the seven more complex figures: the simplest figure, which consists of two concentric circles, gives concentric spectra by both methods of observation.

2nd. *The Diffraction in Shadows*: Fresnel's *Diffraction.*

This is described by Lord Rayleigh as the Diffraction when the source of light is not in focus, or in Jamin's *Cours de Physique* as Diffraction from a Spherical Wave. Light is condensed on a minute pinhole in a thin metal plate; about

a foot from this the diffracting object is placed, and at a similar distance beyond a microscope eyepiece receives the shadows: they may be observed with the eye or projected on the screen of a camera for photography. The rays emerge from the eyepiece parallel, and give an image on the screen which varies in size but not in quality as the screen is moved; infinite changes, however, are made by variations in the relative distances of the source of light, object, and lens or eyepiece.

In order to consider a fundamental principle of importance, take at first more simple apparatus, a convex lens instead of the Huygens eyepiece, and let the object, which is the glass plate with circle combinations, be illuminated with parallel uncondensed sunlight. If the lens is placed at its focal distance from the object, rays emerge parallel and make simple images of circles without any diffraction developments; they are always in focus wherever the screen is placed, and are naturally greater when it is more distant. If the lens is slightly moved from this position in either direction, the emerging rays cross one another and produce interference developments. The figures made when the lens is moved towards the object are inferior in definiteness.

A wave impinging on an edge gives rise to the secondary waves of Huygens, and from each new centre waves spread out in all directions, although the incident light may have a plane front or be a parallel pencil; but a pencil, after emerging parallel from a lens, cannot be considered to have rays striking out obliquely so as to interfere. From this may be imagined the formation of diffraction-rings in a telescope directed to a star when the eyepiece is moved so that the image made by the field-glass is out of its focus; the image is nearly a point of light; while the rays emerged from the eyepiece parallel they would not interfere, but directly they cross one another on emergence they give rise to systems of concentric rings. This phenomenon may be well seen by looking through the eyelashes towards a distant lamp while a fine rain is falling: minute spheres of water fall upon the eyelashes; these short-focus lenses make images of the lamp in front of the eye, and the lens of the eye cannot focus or turn parallel the rays from these points; for a moment a number of fine concentric rings

may be seen, which are constantly evanescent and constantly renewed by fresh-falling specks of water.

There is another phenomenon sometimes revealed when a divergent pencil from a short-focus lens or spherical reflecting surface falls upon the eye in such a way that the eye cannot focus it: a small illuminated disk is seen with fine ramified lines from the centre; in this the eye sees its own blood-vessels: this is easily seen by holding a pinhole near to the eye.

Abundant examples of diffraction may be observed without any apparatus: a distant lamp viewed through the eyelashes on a dry evening appears with many spectral images made by this simple grating: an umbrella can be seen in the same way to act as a rectangular grating. If the eyelids are adjusted to the lamplight, 8 or 10 horizontal exterior bands will easily be produced. Most plate-glass windows have become gratings through the scratching by repeated cleaning.

To return to the main experiment. Stronger effects are produced, of course, when sunlight is condensed on to the pinhole, and in this case the position of the convex lens for giving parallel rays on emergence, or for giving the plain image of the object without diffraction, is not exactly its focal distance from the object, for the divergence of the incident rays has some influence; but there always is such a position. Moreover, the elementary experiment cannot be made with the Huygens eyepiece which is now used, for it has not an external focus. Fig. 8 shows the result of placing the object close up to the eyepiece; the beginnings of diffraction may be seen. Fig. 10 (Pl. III.) shows further developments when the distance is increased, and figs. 9, 11, and 12, 13 (Pl. IV.) give the best effects I have at present been able to produce. In these and in most of the other figures a magnifying-glass is necessary to reveal all the fine detail made by the crossing waves in the shadows. It is clear that if the object is capable of giving spectra passing out with much obliquity, they can never come into the field of the eyepiece by this method of observation; and the eyepiece can never focus them, as it does in the Fraunhofer method, in which case the field-glass brings them to the focus of the eyepiece.

It may be said that the Fraunhofer diffraction gives numerous images of the source of light set in radiating forms which

depend on the shape of the object, while that of Fresnel commonly gives the main shape of the object embellished with fine detail in the spaces between its various parts, and sometimes curiously inverted in its leading features.

A simpler form of this arrangement demonstrates a leading idea in wave theory: parallel sunlight falls upon a convex lens, while the diffracting object is between this and the screen of the camera, which has no lens. Fair diffraction-effects are made when the object lies in the converging rays, and they are better when it is moved to the divergent pencil, but when it is at the focus the plain original figure is reproduced. At the focus the several rays that started from one point of the sun have again all reunited; so that in passing the object, although there are many separate rays, there are no two rays which came from the same original point; and it is only such as these which can produce interference. Here may be seen, in a certain sense, an object and real image on the same side of a convex lens.

Diffraction in a Microscope.

Some notice of this important part of the subject is necessary, although it may not be supported by much illustration or experiment.

It is well, first of all, to be clear as to compound pencils of light which are more or less on the same axes. In a straight line lie A, B, C, D, E, a candle, a device in wire, a convex lens, the images of the candle and the wire: here are two pencils much intermingled: A is in focus at D, but the light here is also impregnated with the shadows of B, which are indefinite and unfocussed, covering a larger area than when focussed at E; and at this point there is also the indefinite unfocussed light of A.

Very often the object on the stage of a microscope diffuses the illumination and becomes a new source of light, and there is no need to consider pencils from the original source of light. But here we will take the case of a microscope with a diffraction-grating on the stage and a small opening in the diaphragm below. Light passes on to the objective from the grating as if from several small apertures ranged side by side, in pairs of corresponding spectra on either side of the central

point, which is white. The objective makes images of these just above itself, and one image of the grating farther on at the top of the tube. If the eyepiece is removed, the eye is able to focus the rays diverging from these images and to see them clearly. But the eyepiece when in its place could not focus such diverging pencils: their light spreads out and fills the field of the eyepiece with general illumination. The visibility of the detail of a finely divided object depends upon shadows thrown by oblique rays. In the case of a grating these oblique rays are so definitely arranged that they go either to the first, second, or higher spectra, and to nothing between. The central pencil comes through the grating normally in parallel rays, and carries with it no appreciable impression of the detail; experiment shows that on placing a stop above the object-glass to cut off the spectral images, the lines of the grating can no longer be seen, although there is still good illumination. Now the first lot of oblique rays, while on their way from the grating to its focus in the eyepiece, take their course through the first spectra; but they are unfocussed as regards the grating at that position just above the objective, covering a larger area than the spectra. In theory perhaps it is conceivable that by stopping out the spectra alone, and not this larger area, there might be a balance of rays which would betray the shadows at the eyepiece. In practice, however, the first spectra are necessary, and perhaps often sufficient, to show the detail: sometimes the higher spectra may be needed to add breadth to the shadows. In fact, from ordinary objects rays of all degrees of obliquity are available to mark the shadows, but from objects capable of diffraction there is no degree of obliquity less than that which tends towards the first spectrum, and the aperture of the objective must not be too narrow to receive these rays.

The question naturally arises, how far it is possible to know whether an image is a reproduction of the general form of the original object or a diffraction modification: so far as my experience goes, the two appearances may be distinguished. Of course in perfect theory, even if an image could be saved from spherical and chromatic aberration, it must suffer something from the waves: the image of a star in a telescope must be enlarged on this account; but there is a great difference

between this falsification and that which arises when the eyepiece is taken out of focus. Fig. 82 is a species of the diatom *Triceratium favus*: it appears as a spotted framework of ill-defined outlines with little groups in the spaces; but the microscope can be adjusted so as to show a well-defined honeycomb with clear hexagonal spaces. In an experiment described above, figures known to be circles are reproduced in their own true form when at one position with regard to a convex lens, and more elaborate ill-focussed forms at positions on either side of the former: it is reasonable, then, to apply this test to distinguish between several images of an unknown original. No doubt the simpler form, which is taken to be the true one, might often be resolved into something more complex by a stronger objective, but this one in its new conditions would again be simple and well-defined amidst its possible diffraction derivatives.

I have not sufficient familiarity with the use of the highest microscope powers, and can imagine that the foregoing tests are not easy, if indeed possible, to apply in such cases; but I suppose that with $\frac{1}{4}$-inch and lower powers there is no need for uncertainty as to the practical form of the original object.

Fresnel's *Diffraction from Geometrical Figures.*

Some interesting details can be observed from the shadows about other figures of Bridge's series. It may be here noted that such objects may be made by drawing rather thick figures in ink upon a card about 1 foot square and reducing by photography to 1 inch square: the negatives so produced give the desired transparent lines on a dark ground: it will be understood that a circle means a circular line of light in distinction to a circular area. The play of light-waves in a shadow gives rise to infinite paradox. The most notorious is Arago's bright spot at the centre of the shadow of a circular disk, with which he astonished Napoleon Buonaparte: squares and triangles turn about and point inwards: needles seem to be split down with the light and the two parts point outwards, the thinner needle having the broader central cleft: lastly, lines inclined to one another have the effect of resolving one another into transverse flakes.

Simple figures may be drawn to give a partial explanation of these curious results.

Fig. *a.*

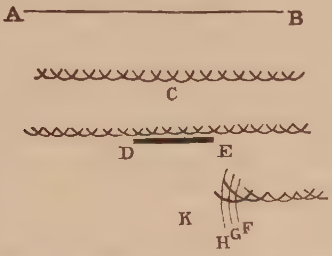


Fig. *a* illustrates the ordinary principle of Huygens, that a wave-front, AB, goes on as a similar wave-front, C, if it be supposed that each point gives rise to a secondary wave: when this strikes upon an obstacle, DE, the waves at the edge, being isolated and not supported by similar waves on the one side, may pass forward so that in directions F, G, H one is always $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$ of a wave-length in front of the other, causing interference or darkness: these are the exterior shadows: also there can be imagined the action of waves from D and E about a position K which gives the interior bands.

Fig. *b.*

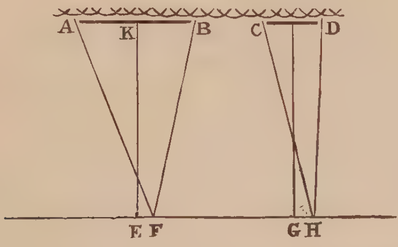


Fig. *b* shows that waves from A and B must always reach E in the same phase, so at the centre there is always light; at F, if the difference of AF and BF $= \frac{\lambda}{2}$, there will be darkness. Now if CH and DH are more nearly parallel than AF and BF are, then, in order to have CH$-$DH$= \frac{\lambda}{2}$, the point H must be farther from G than F was from E, so the smaller

obstacle C D gives the larger bright centre to its shadow. This principle has many consequences: the thinner needle has the broader bright central line, or as a needle is narrowing towards its point the central line will be opening out: at the corner of a square or triangle there will be wedges of light pointing inwards. The enlargement of the image of a star at the focus of a telescope through diffraction is less in a telescope of large aperture, because the waves from the borders of the lens reach the focus at a greater inclination to one another. Again, the length of a wave of yellow light is about 24 millionths of an inch; if the waves from A and B move straight towards one another at K, then the diameter of the bright spot here will be something less than ·000024 inch, too small to be seen, from which it appears that the condition for making visible these phenomena which are connected with such minute quantities is that AF, BF must be nearly parallel: the object must be narrow, or the screen for receiving the shadow must be distant.

The following notes describe the original figures from which the photographs are the shadows:—

14. A circle with four triangular areas.

15. A circular area with four inscribed dark circles.

16. Four circles touching one another; the lines of the circles are gradually made thinner towards the inner part, but they are complete circles.

19. Two ellipses, each through the focus of the other: Arago's spot at the common part.

20. Three pentagons with a small pentagonal area at the common centre.

24 and 29. A square with four external squares at the corners: the latter is the shadow taken at a greater distance from the object; it is covered with fine interference detail.

25 and 28. A square with inscribed square and diagonals.

23, 26, 27, 30. Right-angled triangles put together in sets of four at an acute angle: the detail contains a number of round spots.

31, 39, 40. A chessboard in various states: here again systems of spots are developed.

33. Rows of equilateral triangles, set alternately in successive rows: spots again.

35. Three equilateral triangles with a small triangular area at the common centre.

41. Ten semicircles near to one another and parallel: the feature of this is the broken shaggy brushes at the ends of the semicircles.

43, 44, 45. A parabola lying within another one of smaller parameter; the lines of light being rather thicker about the vertices. In 44 each vertex has opened, turned the broken edge back and taken a small ball into the open mouth: the bands about the axis, which is not drawn in the original, are diffraction creations: the inner parabola is broken into flakes about the vertex by dark cross lines.

42. An hyperbola and its conjugate, twice repeated, without asymptotes. The flaking appearance is noticeable.

46. Two ellipses, side by side: as a larger circle gives a smaller central spot, the curve at the end of the minor axes gives a smaller spot, and that at the end of the major ones a larger spot: the result is the broadening white line in the figure.

47. Five diminishing circles with internal contact, where the lines are made thinner: here are the shaggy offshoots similar to those in 41.

50. Three circles, near, not quite touching. Arago spots and flaking.

48, 49, 51. Arago spots made by small arcs of circles.

52. A square divided into 4 small squares, in each of these a circle inscribed, not quite touching. Not to mention the small detail, this figure suffers the curious inversion of appearing as 4 circles with inscribed squares.

53. Four circles.

54. A square with an external semicircle on each side.

55. Five small circles in a ring touching one another.

56. Two circles, each through the centre of the other.

57. Two unequal circles, near, not quite touching: the larger spot in the smaller circle, and the flaking disturbance of the larger by the neighbourhood of the smaller.

60, 66. Eight small circles in a ring touching one another: in each case there is the bright spot at the centre of each circle, in 66 there is one at the centre of the whole figure:

the formation of the common tangents at the points of contact is not difficult to imagine.

58, 59. The square and triangle which have turned about and point inwards: their sides have been resolved entirely into transverse flakes.

62. A square with thicker lines of light.

61. Not quite a square, a bright square area with an inscribed dark circular area.

65. A pentagon: here, as in several of the figures just above, there is a bright spot at the centre.

Fresnel's *Diffraction from Simple Objects.*

67. The inner part shows Fresnel's interference-lines from a bi-prism, with Grimaldi's fringes at the sides.

68, 69. Narrow slits ; 68, with the broader bands, is from the narrower slit. Both slits are so narrow that the exterior bands, through interference of parts of the same wave from one side, do not come in : the bands are given by waves from the two edges. It may be remarked that in exterior bands there is more indication of colour than in interior bands : the neighbouring rays which form exterior bands can travel almost parallel and make coarser bands with more chromatic dispersion. In the case of light passing on to an aperture, care is sometimes necessary to note which of the two effects is predominant.

70, 81. The eye of a bodkin, one to show interior bands, the other to show the exterior bands : it is not easy to develop both effects at once. The bands within the eye are not interference of waves from the two sides, but the exterior bands of each side: it happens that the breadth is such that a band from one side is superposed on one from the other, so the central line is dark.

79. Four round holes of increasing size. This is a case of exterior bands in the bright spaces ; the centre of the space may be white or not according to the breadth of the hole : also the rings and spots are brightly coloured. These holes varied from $\frac{1}{2}$ to 2 millim. diameter. If a hole be taken $\frac{1}{3}$ millim. diameter, the phenomenon changes and becomes analogous to 68 and 69 : concentric spectra are formed in

the shadow, and there are no rings in the projection of the aperture.

71, 72 (Pl. V.). The points and eyes of ordinary needles: the one with the broadest central line is the shadow of the smallest needle which is made: here the interior and exterior bands both appear.

80. A quartz fibre about ·0005 inch diameter.

73. Wire gauze: each wire gives in shadow 6 dark bands: the spaces give sometimes 2, sometimes 3 bands, showing an inequality of distance. These bands are the exterior bands from the sides of the wire, and give bright colours. The general appearance to the eye is that of a Scotch tartan.

77, 78. Perforated zinc. The former is developed to show the rings in the spaces; these are coloured, and the centre may be white or not according to the size of the holes or their distance from the screen. The latter figure is developed to show the interior bands on the dark parts; they naturally give hexagonal systems. This may illustrate the markings in *Pleurosigma angulatum*, which are regarded as diffraction-bands made by the rows of white spots that form the real structure of the diatom; but, in truth, this case can be realized without the aid of diffraction. Where there are such rows of spots, there must be the lines of shadow between them. The narrowest dark space marked over with diffraction lines has two dark bars separated by a bright central line.

74. Arago's experiment.—The shadow of a threepenny-piece, showing the bright centre. Here parallel sunlight was thrown uncondensed on to the pinhole. The coin was about 18 feet from this, held by a thin wire, and the screen of the camera 18 feet beyond, without any lens. The illumination was not uniform, so that the exterior rings are partly seen: if a lens is used, several rings may be seen around the bright centre, but the complete shadow is too large for the field of view. According to Verdet, Arago employed a circular disk 2 mm. in diameter.

75, 76. Arago's experiment.—Shot fixed on glass; 3 millim., 2 millim., 1 millim. diameter. These were taken with the eyepiece and the ordinary arrangement. In theory there are concentric rings over the whole shadow, but they are more

difficult to see than the bright centre. They appear in fig. 76, and in a strong light may be produced with the larger shot. It should be noticed in fig. 75 (Pl. VI.) and fig. 76 how large a bright centre has been made in several places by specks of dust.

Both with the shot and the threepenny-piece, when they are most accurately arranged, as evidenced by the regularity of the rings, a faint dark spot may be seen at the centre of the white spot, and in some positions of the disk relatively to the eyepiece this opens out into a faint dark ring. At present, I have not been able to ascribe with certainty a reason for this secondary effect, but I think it arises because the source of light is not a mathematical point.

83 (Pl. VI.). Conical refraction; external. In conclusion, I venture to pass the strict limits of the subject by showing the crowning triumph of the hypothesis of wave-propagation. Light through 5 minute pinholes passing in a special direction through a crystal of arragonite emerges in 5 cones of light: the four outer circles show by their imperfection the development of the circle from the two points that are formed by a double-refracting crystal.

Authorities.

Hooke.—Micrographia. 1664.
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Huygens.—Traité de la Lumière. 1690.
Fraunhofer.—Neue Modifikation des Lichtes, &c. Munich, 1823.
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Schwerd.—Die Beugungserscheinungen. Mannheim, 1835.
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Rayleigh.—Encycl. Brit. 9th ed., art. Wave Theory.

Discussion.

Dr. Johnstone Stoney thought the obtaining of permanent records of diffraction phenomena of great importance, and was particularly interested in the photograph showing conical refraction.

Prof. S. P. Thompson said he had never seen diffraction

effects exhibited to an audience so well before. He noticed that in several of the photographs Arago's spot was unintentionally shown to perfection, in the shadows of dust particles.

The PRESIDENT greatly appreciated the fact that the conical refraction photograph had been exhibited for the first time before the Physical Society.

IV. *The Viscosity of Liquids.*

By OWEN GLYNNE JONES, *B.Sc.**

THE object of this paper is to point out a method for the accurate determination of the viscosity of highly viscous liquids. The author would prefer to delay the publication of most of his results till the experiments are completed. The advantages and disadvantages of the method, and its applicability to the determination of viscosities such as those of alcohol and water, are not yet entirely ascertained, but such of these points as have already suggested themselves will be noticed at the end of the paper.

Sir G. Stokes has long since shown that if a sphere moves vertically in an infinite liquid under the action of gravity only, it in course of time assumes a constant speed :

$$V = \frac{2}{9} g a^2 \frac{(\sigma - \rho)}{\mu},$$

where a is the radius of the sphere, σ its density, ρ that of the liquid in which it moves, and μ the viscosity of the liquid. The equation assumes that there is no slipping at the surface of the sphere, *i. e.*, that the coefficient of sliding-friction is infinite. The effect of such sliding-friction would be to modify the equation thus :

$$V = \frac{2}{9} g a^2 \frac{\sigma - \rho}{\mu} \frac{\beta a + 3\mu}{\beta a + 2\mu},$$

where β is the coefficient of sliding-friction.

The method employed by the author consists in the measurement of the downward speed of a sphere of mercury

* Read February 9, 1894.

through the viscous liquid. The amount of mercury used is taken sufficiently small to render possible the measurement, correct to ·1 per cent., of the time taken to pass through a distance of about 25 centim. This also renders legitimate the assumptions that the mercury remains spherical during its motion, and that the liquid in an ordinary burette about 1·5 centim. internal diameter is for purposes of calculation of infinite extent, the motion of such a sphere falling down the burette along its centre line being unaffected by the sides of the vessel.

The burette is jacketed with water kept as nearly as possible at constant temperature. The temperature of water direct from the main was observed last autumn to vary to an extent of only ·03° C. in an hour and a half, and good results were then obtained for the viscosity of glycerine. Within the last fortnight the main temperature has not been nearly so constant, and more has been learnt concerning the variation of viscosity with temperature than the absolute value of viscosity for any definite liquid.

The mass of a sphere of mercury suitable for glycerine would be between ·003 and ·010 grm. This is too small to be determined with sufficient accuracy by direct weighing, with a balance estimating to one tenth of a milligramme. But there is no necessity to determine such a mass directly.

Let a mass of mercury weighing about ·1 grm. be taken and weighed carefully. Let it be divided into about 12 parts, approximately equal to each other. Let the speed of each through a column of very viscous liquid, such as glycerine or clear castor-oil, be determined. Then, using Stokes's result,

$$\mu \propto \frac{a^2}{v},$$

$$\therefore \quad \mu \propto \frac{m^{\frac{2}{3}}}{v};$$

and for a given liquid with constant viscosity

$$m \propto v^{\frac{3}{2}},$$

or

$$m = kv^{\frac{3}{2}}.$$

Hence

$$\Sigma m = k\Sigma v^{\frac{3}{2}}.$$

But $\Sigma m = M$, the whole mass of mercury, and $\Sigma v^{\frac{3}{2}} = V^{\frac{3}{2}}$, the speed that the whole mass would have if it fell as a sphere through the infinite liquid. Perhaps taken as a whole the mercury would not fall as a sphere, and the liquid in the burette would not be as of infinite extent; but by this subdivision such errors are avoided.

We thus get the speed V corresponding to the mass M, and these values can be applied at once in the equation

$$\mu = \frac{2}{9} g a^2 \frac{(\sigma - \rho)}{V},$$

or

$$= \frac{2}{9} g \left(\frac{3M}{4\pi\sigma}\right)^{\frac{2}{3}} \frac{\sigma - \rho}{V};$$

for

$$M = \frac{4}{3}\pi a^3 \sigma, \quad \text{and} \quad a^2 = \pi\left(\frac{3M}{4\pi\sigma}\right)^{\frac{2}{3}}.$$

Moreover the mass of each of the smaller spheres may be easily estimated, since

$$\frac{m}{M} = \left(\frac{v}{V}\right)^{\frac{3}{2}},$$

and

$$m = \left(\frac{v}{V}\right)^{\frac{3}{2}} M.$$

The author has been able in this way to determine the mass of a small sphere (and hence its radius) weighing about ·003 grm., correct to four significant figures. Having thus determined its dimensions, it may be employed for other liquids than that by which these dimensions were obtained.

But the important question suggests itself whether μ is simply inversely proportional to the speed of the falling sphere, or whether, as some maintain, there is a finite coefficient of sliding-friction, and a definite amount of slipping at the surface, which will render the connexion between v and m much more complicated.

This may be tested in a simple manner by observing the speed V for a small mass, dividing it into two parts, and observing the speeds v_1 and v_2 for each part in the same viscous liquid at the same temperature. If there be no sliding-friction,

$$V^{\frac{3}{2}} = v_1^{\frac{3}{2}} + v_2^{\frac{3}{2}}.$$

If sliding-friction exists the effect will be to render the V calculated in the above equation from v_1 and v_2 greater than the observed speed for the undivided mass, the divergence being most clearly shown when the two parts are as nearly as possible equal to each other.

For let us assume that the whole is divided into two equal parts, and that the two speeds observed are V and v_1, corresponding to the two radii A and a.

Now $A^3 = 2a^3$, and A is therefore $= 2^{\frac{1}{3}}a$.

With sliding-friction,

$$V = kA^2 \cdot \frac{\beta A + 3\mu}{\beta A + 2\mu},$$

$$= 2^{\frac{2}{3}}ka^2 \cdot \frac{1{\cdot}26\beta a + 3\mu}{1{\cdot}26\beta a + 2\mu},$$

and

$$v = ka^2 \cdot \frac{\beta a + 3\mu}{\beta a + 2\mu},$$

then the speed calculated for the whole mass from the formula

$$V^{\frac{3}{2}} = v_1^{\frac{3}{2}} + v_2^{\frac{3}{2}} = 2v^{\frac{3}{2}}$$

would be

$$V' = 2^{\frac{2}{3}} \cdot ka^2 \cdot \frac{\beta a + 3\mu}{\beta a + 2\mu}.$$

The ratio of $\frac{V'}{V}$ is therefore

$$\frac{\beta a + 3\mu}{\beta a + 2\mu} \cdot \frac{1{\cdot}26\beta a + 2\mu}{1{\cdot}26\beta a + 3\mu}.$$

Since all the quantities involved are positive, this ratio is evidently greater than unity. As β varies from ∞ to zero, the value of the ratio increases from 1 to a maximum $1 + k$, and then diminishes to 1 again. Thus the divergence between the calculated and the observed speeds does not become greater as the coefficient of sliding-friction diminishes, although this divergence is actually due to the sliding.

Taking the value ·04 centim. for the radius of the sphere, the maximum value of k will be $\frac{1}{44}$, the divergence thus

amounting to about $2\frac{1}{2}$ per cent. The ratio $\frac{\mu}{\beta}$ that gives this result is $\frac{1}{55}$ or $\beta=55\mu$.

If we assume $\frac{\mu\rho}{\beta}$ to have the same value for glycerine that Helmholtz assigns to water, the difference between the observed and the calculated speeds will amount to about 1 in 144 for such spheres ($a=\cdot04$ centim.) as were employed in the investigation.

Various experiments were recently made by the author to test these results; considerable difficulty was experienced in keeping the temperature constant, except when this was 0° C. The tests hitherto made are not regarded as final, and the author would prefer to defer their publication until the series of experiments is complete. But the divergences from the simple law as at first assumed are in all cases within the limits of experimental error, and the method of determining viscosity is not shown to be defective by those test experiments. The temperature difficulty is not so serious in the case of colza-oil, which is one of the liquids experimented upon. The error involved with speeds not exceeding 2 centim. per second through that liquid did not range beyond $\pm0\cdot2$ per cent., and it was as usual to get positive errors as to get negative. Times were estimated only to the same degree of accuracy.

It is possible that sliding may occur if the speed of the sphere M exceeds a certain limiting value. This also is obviated by determining its speed indirectly as $(\Sigma v^{\frac{3}{2}})^{\frac{2}{3}}$, since each individual portion travels so much more slowly.

There is another reason why better results may be expected by this process of subdivision. Mercury is liquid, and a sphere would be deformed if it passed rapidly through the viscous medium. Its speed would be diminished, and it could not be dealt with directly by the fundamental equation connecting μ and V. But by subdivision the masses and speeds dealt with are reduced to such an extent that we may confidently assume the shapes to remain spherical. If the deformation thus produced were appreciable, it could again be manifested by the application of the formula

$$V^{\frac{3}{2}}=\Sigma(v^{\frac{3}{2}}).$$

The calculated V' would be greater than the observed V, the error being of the same sign as that produced by the neglect of any possibly existing sliding-friction.

The error produced by the sides of the burette, which may be so near as to retard the downward motion, is also diminished by the subdivision of the original mass. This also would have the same sign as the other errors specified. The simplest way of testing its existence is to insert into the upper portion of the burette-tube a thin glass cylinder of smaller diameter, and to send down a sphere along the axis of these two concentric cylinders. In the upper portion of its path the sphere is at a smaller distance from the boundary of the liquid than in the lower portion. If the sides actually affect the motion, the speed of the sphere will suddenly change when it enters into the wider portion of the liquid column. If the change of speed is not appreciable, it may be assumed that the diameter of the burette is sufficiently large.

It has hitherto been assumed that we have the means of keeping the temperature of the liquid constant. This is of the utmost importance, inasmuch as the temperature-variation of viscosity is rarely inconsiderable. The viscosity of glycerine changes from 45 to 8 in the range of 16° from 4° to 20° C. At a temperature of 15° C. there is a 10-per-cent. variation per degree change. This implies that to get the viscosity correct to four significant figures the thermometers employed should read to ·01° C.

In the case of olive-oil, using Osborne Reynolds's empirical formula for the viscosity at any temperature between 15° and 50° C.,

$$\mu = 3{\cdot}265\, e^{-{\cdot}0123\, t}.$$

Any such equation, $\mu = \mathrm{A}e^{-kt}$, means that the percentage variation per degree is constant, since its differential equation is

$$\frac{d\mu}{\mu\, dt} = \text{constant}.$$

For olive-oil this constant is ·0123 and the percentage variation is 1·23 for any temperature within the given limits—about one ninth of the change in the viscosity of glycerine per degree. In this case the thermometer should read to $\frac{1}{12}$

of a degree. It is useless to calculate the viscosity of glycerine beyond the fourth significant figure unless there be means for the accurate determination of temperature correct to at any rate ·001° C.

The liquids experimented upon are all bad conductors of heat, and though this is advantageous in that the change of temperature during the course of an observation is not likely to be great, it is a disadvantage in that it takes a long time for the temperature to become uniform throughout the length of the column of liquid. It is quite possible for a difference of temperature of ·1° C. to exist at points in the liquid 1 dcm. apart even though the water has been circulating in the jacket for some minutes.

The temperature can only be expected to be uniform when the circulation has been of constant temperature for about 20 minutes.

But it is of importance to note that small variations in the temperature of the liquid at different points along the path of the falling sphere need not trouble us much if the mean temperature be known. And, similarly, small changes in the mean temperature of the whole during the course of an experiment may be harmless if the mean temperature during the interval be known.

For it can be proved that the speed of flow of a body is the speed corresponding to the mean temperature, however the temperature may vary along the length, if the speed be small and if the temperature-variation be small. Both these conditions are satisfied in the actual experiments undertaken. The inertia of the body is so small that it almost instantly assumes the limiting speed corresponding to the temperature of that part of the liquid through which it is moving.

Let the temperature vary from θ_1 at one end to θ_2 at the other end of the column of length l. Let the corresponding viscosities be μ_1 and μ_2; and let θ and μ be the temperature and viscosity at a point at distance x from the top of the column.

For the small change of temperature assumed,

$$\frac{\mu-\mu_1}{\mu_1-\mu_2}=\frac{\theta-\theta_1}{\theta_1-\theta_2},$$

or

$$\mu = A(\theta_1 - \theta) + \mu_1.$$

Now, if dt be the time taken to traverse the distance dx, the general equation of steady motion gives us

$$dt = \frac{\mu\,dx}{k}$$

$$= \frac{1}{k}\{\mu_1 + A(\theta_1 - \theta)\}dx.$$

Let $\theta = f(x)$, such that

$$\theta_1 = f(0) \quad \text{and} \quad \theta_2 = f(l).$$

Then the time taken for the whole descent

$$T = \frac{1}{k}\int_0^l \{\mu_1 + A(\theta_1 - \theta)\}dx$$

$$= \frac{1}{k}\left\{\mu_1 l + A\theta_1 l - A\int_0^l f(x)dx\right\}.$$

But

$$\int_0 f(x)dx = B, \text{ the area of the temperature-curve,}$$

$$= l\,.\,\theta_m, \text{ if } \theta_m \text{ be the mean temperature ;}$$

$$\therefore \quad T = \frac{1}{k}\{\mu_1 + A(\theta_1 - \theta_m)\}l.$$

But if the temperature had been uniformly θ_m throughout the length l, the time τ would have been

$$\tau = \frac{l}{k}\mu_m,$$

where

$$\mu_m = \mu_1 + A(\theta_1 - \theta_m).$$

Hence $\qquad T = \tau.$

It therefore is only necessary to determine the mean temperature during the interval, and even this is unnecessary if our object is to compare two masses of mercury by their speeds, if they are made to follow each other sufficiently rapidly for the mean temperatures to remain constant in the two cases.

If a variation in temperature along the tube be suspected,

the test is simply to send down a minute sphere and observe its speed for short lengths all along the tube. This speed is constant if the temperature is uniform. In accurate estimations of absolute viscosity the variation in speed should not exceed 1 in 1000 along the tube.

It is probably preferable to read the temperature by a thermometer inserted in the viscous liquid, using a burette of sufficient bore to render the insertion safe. But good results have been obtained with the thermometer in the water-jacket, tied close to the inner tube; this arrangement serves quite well when the water-jacket is fairly constant in temperature.

An observation with glycerine cannot be called good if the temperature-variation exceeds ·03° C., and any that are made with greater variation must be discounted. This conclusion is forced upon one when the periods of falling are taken for the same sphere twenty or thirty times in succession. The curve plotted, connecting the temperature with the time of falling, is a straight line within a range of 1° C., and it is generally found that the points farthest from the straight line correspond to the greatest variation of temperature during the fall. Such a curve supplies us with the means of determining the correction necessary to reduce a speed at one temperature to that at another.

To ensure that the falling sphere shall remain in the axis of the burette, the burette and its jacket are mounted vertically on a stand fitted with levelling-screws, such as that of an ordinary Jolly's balance. The mercury is let down into the burette by a small funnel. The speed is determined by observing the time taken to pass from one mark to another. The distance taken was generally about 50 centim., measured correct to $\frac{1}{2}$ millim., the time taken to traverse that distance being from 100 to 200 seconds. The time was measured with an ordinary watch beating fifths-seconds, or more recently with a Siemens chronograph. In either case the masses of mercury were so chosen that the probable error in a single observation was 1 in 1000.

The following results are for glycerine at temperatures of 18°·28 and 4°·35 C. The former agrees closely with a value obtained by interpolation from Schöttner's results. The

latter would appear to be considerably higher than the corresponding interpolation.

A. Mass of mercury taken, ·03737 grm. This was divided into 12 parts—*a*, *b*, *c*, *d*, &c. . . . *l*.

Greatest variation in temperature during a single observation =·01° C.

Greatest variation in temperature during whole interval =·03° C.

Length of path employed, 50·28 centim.

Ordinary watch used to determine time.

Mean temperature, 18°·28 C.

Mass.	Mean temp.	Time.	Time corrected to 18°·28 C.	$v^{\frac{2}{3}}$.	Mass, m.
	°	seconds.	seconds.		grm.
a	18·27	128·2	128·0	·2462	·003516
b	18·27	121·4	121·2	·2672	·003815
c	18·27	135·2	135·0	·2273	·003245
d	18·27	132·4	132·2	·2346	·003349
e	18·27	117·2	117·0	·2817	·004022
f	18·28	137·8	137·8	·2204	·003146
g	18·29	197·2	197·4	·1285	·001835
h	18·29	148·6	148·8	·1963	·002804
i	18·29	160·8	161·0	·1746	·002492
j	18·29	131·2	131·4	·2367	·003380
k	18·29	136·0	136·2	·2244	·003204
l	18·29	157·8	158·0	·1795	·002564
				Σ=2·6174	Σ=·037372

Taking

$$\sigma=13{\cdot}59 \quad \text{and} \quad \rho=1{\cdot}26,$$

$$g=981,$$

$$\mu=83{\cdot}9\times 12{\cdot}33\left(\frac{\mathrm{M}}{13{\cdot}59\times 2{\cdot}617}\right)^{\frac{2}{3}},$$

$$=10{\cdot}69 \text{ for the mean temperature } 18^{\circ}{\cdot}28 \text{ C.}$$

B. Mass of mercury taken, ·1096 grm. This was divided into 10 parts.

Greatest variation in temperature during a single observation =·07° C.

Greatest variation in temperature during whole experiment $=\cdot 22°$ C.

(This result is therefore not so reliable as the previous one.)

Mean temperature, 4°·35 C.

Siemens chronograph employed to measure time.

Length of path employed, 50·28 centim.

Mass.	Mean temp.	Time.	Time corrected to 4°·35 C.	$v^{\frac{3}{2}}$.	Mass, m.
	°	seconds.	seconds.		grm.
a	4·30	111·6	111·1	·08401	·01080
b	4·32	101·4	101·1	·09680	·01244
c	4·45	116·1	117·2	·07751	·00996
d	4·28	123·1	122·4	·07265	·00934
e	4·41	99·8	100·4	·09782	·01257
f	4·48	150·5	152·3	·05236	·00673
g	4·23	89·75	88·77	·11760	·01512
h	4·32	107·4	107·1	·08876	·01141
i	4·40	108·3	108·8	·08667	·01114
j	4·50	114·3	115·9	·07885	·01014
				$\Sigma=\cdot 85303$	$\Sigma=\cdot 10965$

Thus $$\mu_{4°\cdot 35\text{ C.}}=46\cdot 27.$$

The value of $\Sigma v^{\frac{3}{2}}$, when no corrections are made for temperature, is found to be

$$V^{\frac{3}{2}}=\cdot 85434,$$

which agrees fairly well with the above result.

The temperature-variation of viscosity is considerable. It is of practical importance in certain applications, as, for example, in the use of viscous liquids for lubricating purposes. A special piece of apparatus was used by the author to investigate this variation for different liquids.

It consists of a glass tube closed at each end, with a lateral tube fixed at right angles to its middle portion and forming an outlet. The vessel is filled with the liquid and a very minute sphere of mercury is inserted. A thermometer passes through a stopper that encloses the liquid, the bulb of the thermometer reaching just up to the main tube. The vessel is immersed in a copper water-bath so that the main tube is vertical and the thermometer horizontal. The thermometer

passes out of the bath through a small stuffing-box on one side. A window of glass is inserted in the opposite side of the bath through which the motion of the mercury may be observed. As soon as the mercury has fallen to one end of the tube, it is reversed and the mercury allowed to fall down to the other end again.

The thermometer reads the temperature of the viscous liquid at its middle, this being very nearly the mean temperature. The bath is heated slowly to about 50° and then allowed to cool, during both of which operations the times of descent of the mercury from one fixed mark to another are carefully noted through the window, the water being stirred constantly. The times of descent are proportional to the viscosities, if a slight correction be made in each case for the variation in the densities of the mercury and the liquid, with change of temperature.

To illustrate the nature of this correction let us take the case of glycerine,

$$\mu = k\,\frac{(\sigma-\rho)}{\sigma^{\frac{2}{3}}}\,\mathrm{T},$$

where T is the time of falling, and σ and ρ are the densities of mercury and glycerine respectively.

If s and r are the corresponding coefficients of expansion,

$$\mu = k\,\frac{(\sigma_0-\rho_0)-t(\sigma_0 s-\rho_0 r)}{\sigma_0^{\frac{2}{3}}}\,(1+\tfrac{2}{3}st).$$

This is approximated by taking $s=\dfrac{1}{5500}$ and $r=\dfrac{1}{1000}$,

$$\mu = \mathrm{K}(12{\cdot}34+{\cdot}0002\,t)\mathrm{T}.$$

The correction is therefore unnecessary between 0° and 30°, and is only 1 in 1200 from 30° to 70°. For ordinary temperatures it may therefore be neglected.

With a relative viscosity-curve thus obtained, and with one good absolute determination, we have the means of calculating the absolute viscosity at any temperature along the range.

Conversely, we have the means of estimating small masses of mercury by their speed of flow through glycerine or any

other known liquid at a known temperature, or of estimating the mean temperature of the liquid.

For example, if a column of liquid be heated from above till its state of temperature becomes constant without the aid of convection, we can here determine the temperature-curve along its length, and in fact employ Forbes's method to determine the thermal conductivity of the liquid at various temperatures.

In concluding this description of the methods now being employed for the determination of absolute and relative viscosities, it may be well to remark on the advantages and disadvantages of these methods that have already manifested themselves. In the first place we are dealing with steady motions, and are able to investigate the phenomena attending a constant rate of shearing in the liquid much more satisfactorily than if we observe an oscillating motion such as that of Coulomb's disk or Helmholtz's sphere. The existence of sliding-friction can be directly tested, not only between mercury and the more highly viscous liquids, but also between any two liquids that do not dissolve each other. Thus spheres of water may be used with nearly all the fixed oils, and spheres of oil of cloves (density 1·0475) or of oil of myrrh (density 1·0189) may be used with water.

The apparatus is simple and inexpensive ; results may be rapidly obtained when a few standard mercury spheres are preserved. They should be kept in a sample of the viscous liquid to be tested. If a sphere breaks, the pieces should be washed in water and reunited on hard pressed blotting-paper. The quality of oils is often tested by their viscosity, and special viscosimeters on the capillary-tube principle of Poiseuille are used for the purpose. A time-reading through a sample of the oil with a standard mercury sphere offers an expeditious way of testing. If the oil is thin and the mercury falls too fast, a calibrated water sphere may be used instead. A sphere of water of 1 millim. radius, coloured with eosin to be clearly visible, travels at the rate of one inch per hour in castor-oil at 8° C. ; and here in parenthesis it may be added that we have by far the simplest method of observing the time-integral of temperature for small ranges.

The general method cannot be employed for opaque liquids,

as we wish to observe the falling sphere; but it is probable that with a little ingenuity this difficulty could be overcome if the opaque liquid presented itself for examination. The small inertia of the falling sphere, advantageous as it is in exhibiting the slightest variations of temperature, is a serious objection if small solid particles are held in suspension in the liquid. As a rule these particles will avoid the small sphere and not touch it; but in the event of contact occurring there is the likelihood of a permanent union between the two and of the particle being dragged down with the sphere, with consequent loss of speed of the latter. Hence clear liquids must be used.

The author wishes to acknowledge his obligations to Prof. Henrici for very kindly rendering available the apparatus of his laboratory for the needs of the above experiments.

Discussion.

Prof. Everett, in a written communication, suggested that the motion of the liquid sphere be checked by using beads of quartz or glass.

Lord Rayleigh pointed out in a letter that the formula employed related to a solid sphere, and thought it not legitimate to use it for liquid spheres, for the tangential forces at the surface would set the interior liquid in relative motion, and modify the resistance experienced. He also thought the existence of a finite coefficient of sliding friction between two fluids an impossibility.

Mr. Watson said temperatures might be kept constant for days together by Ramsay and Young's vapour-jacket.

Dr. Sumpner thought the surface-tension of such small spheres of mercury was so very large that they would act practically like solids. The want of solidity might be of importance when the two liquids were very nearly alike in density and other properties.

Mr. Blakesley said that at high velocities the falling sphere might acquire a palpitating motion in addition to the gradual descent, and this might introduce errors.

Prof. Perry considered that the experiments on the velocity of a small sphere, and those of the two parts in

which it was divided, which showed that

$$V^{\frac{2}{3}} = v_1^{\frac{2}{3}} + v_2^{\frac{2}{3}},$$

proved the simple formula used to be correct.

Mr. Boys inquired if any tests had been made on the constancy of dimensions of the spheres used. He would expect that in the case of water and oil, for example, that mutual contamination would take place. Speaking of the indirect method of determining the masses of small spheres, he thought direct weighings might be made, for, as the President and Prof. Poynting had shown, the balance might be immensely improved.

Prof. S. P. Thompson suggested that small globules of aluminium or slag might be used.

Dr. C. V. Burton thought Lord Rayleigh's criticism important, and that large corrections might be necessary. He failed to see how the large surface-tension mentioned by Dr. Sumpner could prevent internal circulation.

Mr. Trotter said Lord Rayleigh's point might be tested by using a sphere of oiled wax.

Mr. Boys mentioned that Lord Rayleigh had shown in the case of soap rings that variation of surface-tension, due to stretching or compression, produced stability. The same phenomena would probably retard internal circulation.

The President said there was little doubt that internal circulation as mentioned by Lord Rayleigh would modify the velocity.

In his reply Mr. Jones said he could not imagine how, in pure liquids, internal motion in the falling spheres could be set up. In answer to Mr. Boys, he had found slight changes in the masses of the water spheres after being used many times, but this was a question of days. During an ordinary series of observations the dissipation was too small to be observed.

After the meeting had been adjourned Mr. Boys and Dr. Burton considered the question of internal circulation, and the latter pointed out that with perfectly liquid spheres there would be infinite slip, and the coefficient of sliding friction β would be zero. The velocity of descent would therefore be $\frac{3}{2}$ times that given by the first equation.

V. *On the "Magnetarium." By* H. Wilde, *F.R.S.*

(Abstract.)

Mr. Wilde exhibited and described his "Magnetarium" (March 9, 1894).

This consists of a hollow geographical globe wound all over the inner surface with insulated wire in planes parallel to the equator. Within this globe is a sphere wound with wire on its surface, and having its axis inclined at 23½° to that of the outer pole. By means of epicyclic gearing the spheres can be made to rotate at slightly different rates. When electric currents of suitable strength are passed through the two windings, the magnetic condition of the earth can be imitated, both as regards distribution at any epoch and the secular variations. A better result was obtained by putting sheet iron over the land areas, and a still closer approximation by using thin iron over the water areas. A magnetic chart and tables giving the magnetic elements at various places for different epochs were shown. The author mentioned that recent observations by the United States Survey at Ascension Island, and by Prof. Thorpe in Senegambia, had confirmed results obtained by the magnetarium.

Discussion.

The President said he had tried the apparatus, and found the Siberian oval closely imitated. The secular variations at Greenwich were also well shown. In South America the approximation was not so good.

In reply to a question by Mr. Blakesley, Mr. Wilde said the present position of the pole of the inner sphere was 84° W., 67° N.

VI. *A new Electrical Theorem.* *By* THOMAS H. BLAKESLEY, *M.A.**

THE very short paper which I shall read to the Society contains the account of a Theorem which, though admitting of easy proof, appears, so far as my inquiries have gone, to have hitherto escaped notice.

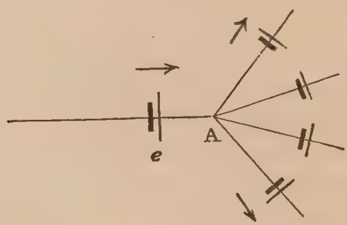
In order to state the matter briefly, it will be well to adopt the following definition :—

If in any system of conductors, however reticulated, two or more modes of disposition of sources of electromotive force produce in every part of the network the same current, such systems of disposition are called *equivalent systems*. Then the theorem is as follows :—In any system of conductors, possessing seats of electromotive force at any number of points, if any of these sources be supposed to move continuously along the various bars of the conducting system, and, where a point of junction is encountered, each to become a seat of the *same* electromotive force in *each* of the newly encountered bars (of course leaving the resistance of the source behind), then the disposition at any moment is equivalent to that at any other moment, and therefore to the original disposition. [Of course the direction of the electromotive force must be carefully maintained the same: if it is towards the knot before crossing it, it must be away from the knot after passing it.] The proof need only refer to the passing of a knot point, for no one will doubt that if the sources only move in an *unbranching* portion of the conductor the currents in different parts of the system will remain the same.

Let therefore the source e approach the point A at which its path splits into n other ways. In each of the n bars suppose a source e inserted as directed, then these n alone must be equivalent to the single source before reaching A ; for if the n sources are reversed, the current due to these sources in every portion of the system is reduced to zero. The reversed n sources would therefore alone produce currents in the system

* Read February 23, 1894.

equal numerically, but opposite in direction, to those produced by the single source. Hence it follows that the n sources (not reversed) will produce the same current as the single source.



The principle of the superposition of currents enables us to apply this result to each source of the system, and therefore to prove the truth of the theorem in its complete generality.

In equivalent systems, since the current in every part remains the same, the total power expended remains the same; and equivalent systems might have been defined as those which produce the same expenditure of power in each part; and therefore the total power of the sources remains the same.

From the point of view of Kirchhoff's theorem $\Sigma e = \Sigma . rc$ for any closed path in a network, the above general theorem may seem to some minds even plainer and more easily proved than on the method of demonstration which I have employed. For it is plain from the method of derivation, that if a seat of electromotive force exists in any closed Kirchhoff path it can never leave it; and if in the movement of the sources one of them approaches the closed path under consideration, at the encounter it becomes in that path two equal sources acting in opposite directions.

If, therefore, Σe remains the same for any path and r remains the same for every part, then obviously c must remain the same for any portion of that path, and therefore for every part of the network.

The following propositions flow immediately from the main proposition :—

(1) If a closed continuous surface contains any region of any network, and some bar which cuts the surface contains, or by derivation as above can be made to contain, the seat of an

electromotive force, then that source can be done away with without disturbing the currents in any portion of the system provided that in the other bars which cut the surface sources of electromotive force be inserted of equal value but of opposite direction as regards inside and outside of the surface; for it is clear that such sources would result from the migration of sources in one direction.

(2) If two systems of electromotive forces are equivalent, one may be derived from the other. For if system A is equivalent to system B, and we suppose $\frac{A}{2}$ to represent a distribution identical with A as regards positions, but of half the electromotive force in any case, then $\frac{A}{2}+\frac{B}{2}$ is equivalent to A or B alone. Now if any Kirchhoff path containing a source from $\frac{A}{2}$ does not also contain a source from $\frac{B}{2}$, then Kirchhoff's law would be outraged; for the sum of the electromotive forces in that path would be only half what they are from A alone, whereas the currents and resistances remain the same. Hence for every Kirchhoff path there must be equal sources from each system. Either system may now have its elements moved up to those of the other system; any resulting side branchings will be the same (though differing in sign) whether derived from $\frac{A}{2}$ or from $\frac{B}{2}$, and must necessarily produce no effect by themselves, because if we consider $\frac{B}{2}$ to be reversed, the whole effect of $\frac{A}{2}$ and $\frac{B}{2}$, *now in the same bars*, will be zero.

VII. *The Attachment of Quartz Fibres.* *By* C. V. Boys, *A.R.S.M., F.R.S.**

Members of the Physical Society may remember that in 1887 † I described a method of making fine fibres of glass and other materials, but especially of melted quartz, which latter had properties of great value, rendering them more suitable for experimental work of combined delicacy and accuracy than those of other known materials. Experiments made since by others as well as myself have further shown that for delicate work of the highest degree of accuracy they are essential.

The method of fastening them, however, at their ends to the pointed end of the torsion-pin at the top or of the suspension below by shellac varnish, or better by melted shellac, is apt to give rise, more especially if the fibre is unskilfully laid in place so that it is twisted round the point, to a slow creeping of the point of rest due to slow changes in the shellac. This, except for the first few days, can hardly ever be of an amount to seriously affect any observations; in fact I have made many observations of the effect of gravitation between small masses with fibres so fastened of a great degree of accuracy, besides those with the radiomicrometer, pocket electrometer, &c., without any inconvenience, yet I have felt that some method of attaching them which would be less likely to hold the fibre by a part in a state of torsional strain or of flexure would be preferable. If the part of the fibre held could certainly be in its natural position and state with respect to the rest, then, even if the fastening should fail to be as perfect as a true weld, any resulting change of zero should be small compared to that observed where the portion held is twisted or much bent.

The process of silvering, electro-coppering, and soldering is an obvious one, but it is not so easily carried out with a fair degree of certainty and in a manner which is convenient of application, as might be expected. My experience of last

* Read February 23, 1894.

† Proc. Phys. Soc. vol. ix. p. 8.

autumn has enabled me to perform the process in a series of operations, each simple enough, and, as far as I am able to test it in the apparatus with which I am now measuring the Newtonian constant of gravitation (which I may say is of unusual delicacy), with perfect success. In this case the fibre is necessarily stretched to not far from its breaking weight, and it is in such cases that the stability of the fastening is most severely tried.

The first thing I found was that it was a mistake to solder the fibre to the torsion-rod and to the suspension directly. The difficulty of the manipulation is great and a change of fibre is very troublesome. The preferable plan is to solder the ends of the fibre to little tags of metal so small and light that they may be picked up by the fibre from anything on which they rest without risk of snapping the fibre at the point of junction. These tags, which are conveniently made of copper-foil, five millimetres long and one millimetre wide about at the wide end, tapering nearly to a point, can afterwards be fastened to the torsion support and the suspension by shellac varnish or by melted shellac, and now the enormous surface and the stiffness of the foil is sufficient to prevent any trouble from the causes to which reference has already been made.

These tags might also for some purposes, either or both of them, be made of T-form to hang in a pair of V's, and so dispense with cement altogether, and allow of the easy interchange of suspensions or of fibres, but I have not myself employed such a form.

The following operations are those which I have found to answer :—

1. Select a fibre of the right diameter to give the desired torsion. Since the torsion depends on the fourth power of the diameter, a small change in the diameter makes a fourfold change in the torsion, and great accuracy of measurement is needed where an exact torsional rigidity is required. Cut off a piece from two to three centimetres longer than will ultimately be required.

2. Fasten to the extreme ends of the fibre, with melted shellac, little weights of gold or platinum heavy enough to pierce a liquid surface.

3. Hang the fibre over a fixed round horizontal rod of wood, 1 centimetre in diameter or thereabouts, so that the little weights hang side by side, and lift up from below a little glass of strong nitric acid, so as to wet and clean the fibre well above the final points of attachment. The vessel must be wide enough to prevent capillarity from drawing the fibres to one side, or it must be brimful so that the surface is convex, which with nitric acid is objectionable. The vessel must be moved both upwards and downwards past the place at which the weights pass through the surface very rapidly, practically with a jerk; otherwise the weights will be drawn together by capillarity, and the fibres will get twisted, or capillarity will give trouble somehow. With the rapid movement the little weights hardly acquire any pendular motion.

4. After a minute or two do the same, but to a slightly greater depth, with water which may be distilled.

5. When the acid may be supposed to be washed off, immerse in the same way in Rochelle salt silvering-solution (Kohlrausch, 'Physical Measurements,' p. 115).

6. Wash as in 4.

7. Fill a glass with the copper solution that is employed in electrolytic measurements of current, *i.e.* not saturated, and slightly acid. Dip the extreme point of the positive wire from a single cell into the liquid, and with a clean smooth negative wire take the hanging ends, one at a time, and having made the contact outside the glass by resting the upper part of the silver coat upon the wire, let down into the solution, keeping the fibre in gentle movement on the wire and making it dip more and less in the liquid. In a few seconds the little weight will be bright red, and the immersed portion of the silvered coat will be bright red also. The silver coat has sufficient resistance to prevent unduly rapid deposition. Do the same to the other end.

8. Cut off to length, allowing about 5 millimetres at each end for the junction. Take tags of copper-foil three or four centimetres long and three or four millimetres wide, tapering to a point, and having tinned the pointed end of each with the minimum of solder, again wet with chloride-of-zinc solution. On the wet surface lay the coppered end, taking

care that it points in the right direction. Capillarity will now hold it. Rapidly heat the copper to the melting-point by holding a point about one centimetre from the narrow end over a minute flame. The solder will flash and envelope the coppered fibre. Cut off the tag of the desired length, holding the metal by the tag with a pair of pliers and not by the heavy end.

9. Wash in boiling water as in 4 to remove chloride of zinc. The fibre is now attached, but the protruding silver and copper give a want of definiteness in the place of attachment.

10. Dip up to the point of the tag in melted beeswax, following the precautions given in 3, but the two tags may be more conveniently dipped separately.

11. Dip up to the top of the copper and silver in strong nitric acid.

12. Wash in boiling water, which removes acid and beeswax and leaves the fibre ready for use.

13. If it is required to conduct electricity, as for instance to keep the needle of a quadrant electrometer electrically connected with a battery, the whole may now be silvered in a long tube and washed, otherwise it will insulate most perfectly. It may be mentioned here that for the most delicate possible electrometer, as I found in my experiments on the pocket electrometer, it is useless to expect to find any stability where a liquid surface is pierced. The only method of communicating with the needle is through a silvered quartz fibre. Owing to the insulating quality of a clean quartz fibre, delicate experiments are apt to be disturbed by unintended electrification of the suspension, and this may still remain after means have been employed to prevent it, for mere metallic contact between different metals leaves the surfaces in effect at different potentials, depending on the metals used, and, as I showed, in an idiostatic instrument the disturbance due to platinum and zinc is many hundred times the least that can be detected.

I have sought to reduce this form of error by either or both of two methods. In the first I make the inside of the chamber surrounding the suspension a figure of revolution, the axis being the line of the fibre; in the other, when possible,

I make the surfaces of the suspension and of the enclosure one and the same, preferably electro-gilt.

The first method in very small instruments also in the main avoids what can no longer be safely neglected, as it has hitherto nearly always been, viz. gravitational attraction.

There is one more point which may be of some interest. If an unsilvered quartz fibre is threaded through a small hole in a thin metal plate, stretched by a suspended weight, and the hole is then wetted with chloride of zinc and soldered up, the fibre will, after washing off the fused chloride of zinc, pull out, leaving a hole fine and beautifully circular.

It is unnecessary to say more than I have already done, on more than one occasion, on the necessity for making the free space round a suspension in any instrument of extreme delicacy as small as possible and enclosing it by massive metal, itself protected from outside heating and cooling by a non-conducting cover, such as I have in the radio-micrometer; otherwise the convection currents set up in the free space will blow the suspension about, and produce vagaries which might be easily attributed to the fibre or its attachment. The disturbances due to this cause are apt to be much greater than anyone would at first imagine, and the small trouble spent in avoiding them in the manner indicated is well rewarded.

With regard to the manipulation with fine fibres, I have already pointed out that the darkness inside a drawer just pulled out, if the operator is sitting at a table in front of a window with a good light, is such that fine fibres can readily be seen upon it as a background. No velvet or smoked surface or artificial blackness of any kind is comparable with it. On such backgrounds fine fibres are to all intents and purposes invisible. What is in many respects preferable to the dark background, at least in certain operations, is a plain looking-glass lying on the table. Fibres resting upon it become intensely brilliant and visible, provided the eye is so placed as not to see the sky light itself reflected from the mirror. One method of making the fibres very easily visible without influencing their torsion, is to smoke them with burning magnesium or arsenic. I do not suggest arsenic, but I mention it because of the very beautiful effect I once observed, after destroying all life in a small hot-house by burning a

large quantity of bengal fire in which orpiment is a considerable constituent. All the spider-webs remained perfect with the spiders in their places as though alive, and the webs were of a dazzling white but perfect in form, undragged by the weight of the white arsenic upon them, thus contrasting strongly with the catenary distorted webs so much admired in frosty weather. It was this observation that suggested the magnesium smoking.

These last few points hardly come directly under the title of this paper, but I thought them worth adding as bearing upon the successful design of apparatus in which the full limit of delicacy and accuracy obtainable by the quartz fibre may be obtained, and upon the practical details of its treatment.

Discussion.

Mr. Inwards asked if the shellac used to secure the tags was melted or dissolved.

Mr. Blakesley inquired if silvering fibres did not destroy their perfect elasticity.

Dr. Sumpner wished to know if any data as to the relative torsional rigidity of silvered and unsilvered fibres had been obtained, and if the electric resistance of silvered fibres had been determined.

Mr. Watson said silvered fibres had been successfully used in electrometers. As regards their torsion, he had found it differ from day to day, and the resistance varied enormously.

In reply to a question, Prof. Boys described the exact process of soldering the coppered fibre to the tags. As to the torsion of silvered fibres, he would not expect much increase as the film was very thin. He also thought the elasticity would not be destroyed, for silver and gold make very good torsion wires.

VIII. *A Method of finding the Refractive Index of a Liquid; applicable when the Liquid is not Homogeneous.* *By* T. H. LITTLEWOOD, *M.A.**

Apparatus required.

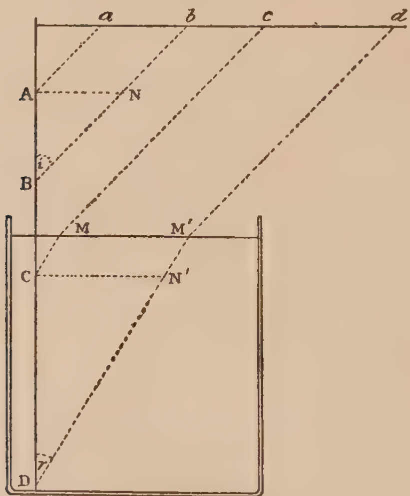
THE chief piece of apparatus required for the method is a telescope with fixed wires in the eyepiece, arranged so as to be capable of motion along a horizontal scale, without changing its inclination to the vertical. The horizontal motion can be measured either by a vernier or by a micrometer-screw.

The liquid whose refractive index is to be determined is placed in a glass vessel about 3 feet away from the telescope, and with its level slightly below that of the telescope.

A scale of glass (or some material not acted on by the liquid) is placed vertically in the liquid and illuminated by monochromatic light. The position of the telescope on the horizontal scale is then taken when observing the various divisions on the vertical scale. From these readings the index of refraction can be ascertained.

First case.—When the liquid is homogeneous (fig. 1).

Fig. 1.



* Read February 23, 1894.

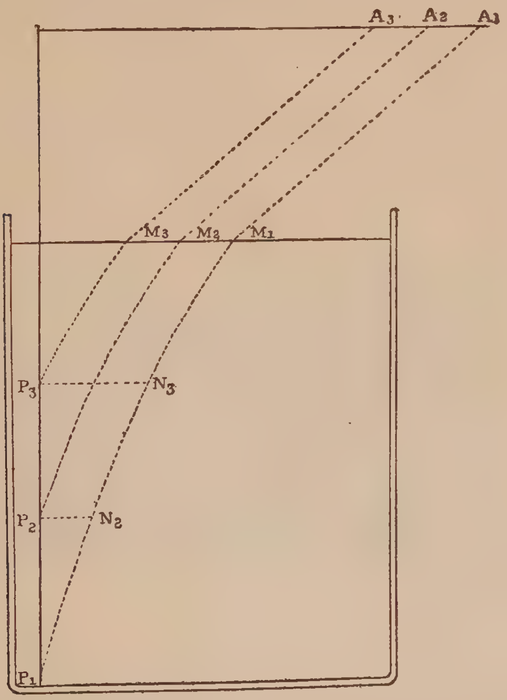
Suppose A, B are two points on the vertical scale out of the liquid ; C, D two points within the liquid. Suppose a, b, c, d are the positions of the telescope on the horizontal scale when observing these points. Then Aa, Bb are parallel ; and if CMc, $DM'd$ are the directions of the axis of the pencil of light from C and D, Mc, $M'd$ are parallel, and also CM, DM′ are parallel. Draw AN and CN′ parallel to $a\,b\,c\,d$. $AN=ab$, $CN'=cd$.

$$\therefore\ \tan ABb=\frac{ab}{AB},\ \text{and}\ \tan CDM'=\frac{cd}{CD}.$$

$$\therefore\ \mu=\frac{\sin i}{\sin r}=\frac{\sin ABb}{\sin CDM'}=\frac{ab}{cd}\times\sqrt{\frac{cd^2+CD^2}{ab^2+AB^2}}.$$

Second case.—When the liquid is not homogeneous (fig. 2).

Fig. 2.



The liquid is supposed to have the same density at the same depth, which must be the case if it is at rest. The path of

the axis of the pencil of light from any point on the scale which enters the telescope must be a curve.

Suppose P_1, P_2, P_3 various points on the vertical scale, and A_1, A_2, A_3 the corresponding positions of the inclined telescope when observing them. Suppose M_1, M_2, M_3 are the points where the axis of the pencil cuts the surface of the liquid, and draw P_2N_2, P_3N_3 parallel to the surface. Then it is clear that, since M_1A_1, M_2A_2, M_3A_3 are parallel, P_2M_2 is the same curve as N_2M_1 moved horizontally, parallel to itself, through the distance P_2N_2 or A_1A_2. P_3M_3 is the same curve as N_3M_1 moved through the distance P_3N_3 or A_1A_3. Similarly for other points.

Hence, by taking a number of points on the vertical scale and finding the corresponding positions of the telescope, and then plotting a curve having the vertical distances from the lowest point for ordinates, and the observed distances through which the telescope has been moved from its first position as abscissæ, we can construct the path of the axis of the pencil through the liquid.

A previous observation of different points on the scale, before the liquid is poured into the vessel, gives the inclination of the telescope to the vertical, as in the first case. By measuring the inclination to the vertical of the tangent to the curve obtained, we can determine the refractive index at the various points of the liquid.

Assuming the curve for a short distance to be a straight line, the index of refraction of the layer of liquid between any two points can be calculated as in the first case, and a similar formula will be true.

Discussion.

The President said the method described was a novel and interesting way of picking out the layers of liquid of different refracting power.

IX. *On a new Harmonic Analyser.*

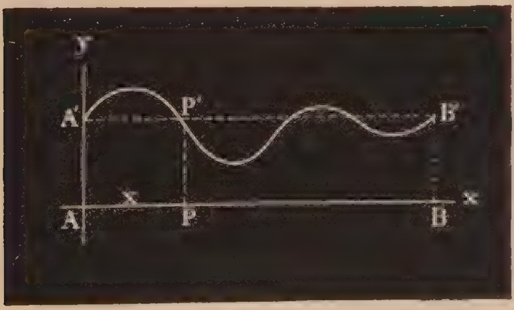
By Prof. O. Henrici, *F.R.S.**

§ 1. According to the theory of Fourier's Series any function y of x can, under certain restrictions, be expanded in a series progressing according to cosines and sines of multiples of x.

This function may be represented graphically by a curve, x and y being taken as rectangular co-ordinates, or it may be defined by aid of such a curve.

Anyhow, we shall suppose this curve given, and also that it extends from $x=0$ to $x=c$ (fig. 1). For this interval the curve may be drawn perfectly arbitrary as long as it gives for every x one single finite value of y. This implies that if a point moves along the curve the corresponding value of x always increases. The curve may, however, be discontinuous, so that for a particular value of x the ordinate changes suddenly from a value y_1 to a value y_2, as from C to C′ in fig. 2. There may be any finite number of such discontinuities. For our purposes it is necessary to make the curve continuous by joining the two points C′ and C by a straight

Fig. 1.

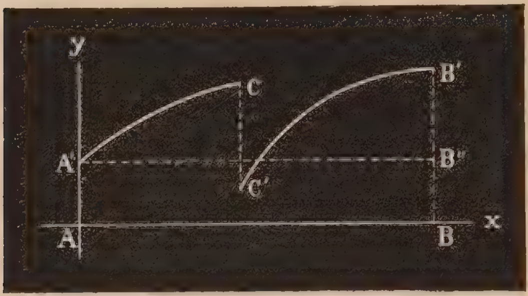


line. If the curve represents a periodic phenomenon with period c, then the ordinate for $x=c$ will, as a rule, equal the initial ordinate for $x=0$ (as in fig. 1). The curve when

* Read March 9, 1894.

repeated along the axis of x will therefore be continuous. Otherwise there will be a discontinuity as at B in fig. 2. In this case also the curve has to be continued from its end point

Fig. 2.



B′ along the last ordinate to a point B″ which has the same ordinate as the initial point A′, so that the line A′B″ is parallel to the axis of x.

We can now express the equation to the curve in the form of a Fourier Series,

$$y=\tfrac{1}{2}A_0+A_1\cos\theta+A_2\cos 2\theta+\ldots+A_n\cos n\theta+\ldots$$
$$+B_1\sin\theta+B_2\sin 2\theta+\ldots+B_n\sin n\theta+\ldots$$

where $\theta=\dfrac{2\pi x}{c}$.

The absolute term $\frac{1}{2}A_0$ equals the mean ordinate of the curve, and can therefore be determined by any planimeter. It is the object of the Harmonic Analyser to determine the other coefficients. Their well-known values are

$$A_n=\frac{1}{\pi}\int_0^{2\pi} y\cos n\theta\, d\theta\,;\quad B_n=\frac{1}{\pi}\int_0^{2\pi} y\sin n\theta\, d\theta.$$

The Analyser is therefore an integrator.

If the paper with the curve be wrapped round a cylinder, the ordinate y falling on the generating lines or edges, the axis of x along a circumference, then the curve will run back in itself and form one continuous line, provided the circumference of the cylinder equals the base c. That edge which

passes through the initial point A' of the original curve may be called the zero-edge.

Suppose the cylinder to lie horizontal with the zero-edge at the top, then our angle θ will be the angle through which the cylinder has to be turned in order to bring that point P to the top which corresponds to any given x. Each edge contains one point on the curve, excepting in case of a discontinuity where a finite length of the edge belongs to the curve.

§ 2. The first instrument of this kind was constructed by Lord Kelvin (Proceedings Roy. Soc. vol. xxiv., 1876). Since then several others have been devised. With regard to these I may refer to my article "Ueber Instrumente zur harmonischen Analyse" in the Catalogue prepared by Prof. W. Dyck of Munich for the Mathematical Exhibition which was held last summer in Munich, and also to the descriptions in the Catalogue of the various instruments exhibited.

These instruments differ essentially either in the manner in which the trigonometrical factor is introduced, or in the arrangement by which the actual integration is performed. Lord Kelvin uses for the latter purpose his brother's disk-globe and cylinder integrator, whilst a simple harmonic motion introduces the trigonometrical factor. Sommerfeld and Wiechert * of Königsberg make the cylinder on which the curve is drawn rotate about an axis perpendicular to that of the cylinder, and thus avoid the simple harmonic motion, which is always a drawback, as it introduces a great deal of friction. Both instruments are also large and heavy, practically fixtures in the room where they are used.

§3. Clifford has given a beautiful graphical representation of Fourier's Series, which I knew more fully from personal communication than from the short paper published in vol. v. of the Proceedings of the Lond. Math. Soc.

His result may be stated thus:—"If the curve to be analysed be stretched out in the direction of the x to n times its base without altering the y, and then wrapped round a cylinder with circumference c so that it goes n times round, then the orthogonal projection of this curve on that meridian

* See above Catalogue, p. 274.

plane which passes through the zero-point of the curve will enclose an area which is proportional to the coefficient B_n. In the same way A_n is got by aid of a plane perpendicular to the first."

It was this theorem which led me to the construction of an Harmonic Analyser. It can easily be put in the following form. Suppose the cylinder placed with its axis horizontal and the tangent plane to its upper edge drawn. This edge cuts the curve in n points. Let P be one of them. If now the cylinder be turned, and if at the same time the tangent plane be moved in its own plane in a direction perpendicular to the edge of contact, the point P will trace a curve on it. This plane will be the same as Clifford's curve in case the motion of the tangent plane is simply harmonic, completing one period for each rotation of the cylinder. The curve will be completed after n rotations of the cylinder.

The same curve will be traced if the original, unstretched, curve is wrapped (once) round the cylinder, whilst the tangent plane completes n periods of its simple harmonic motions for one revolution of the cylinder.

We thus get in a fixed plane a curve whose area equals, in some unit, the coefficients A_n or B_n, and this area can be determined by an ordinary planimeter. The curve, of course, need not be drawn out, as long as the tracer of the planimeter is always at the point P it will describe the curve.

This can easily be realized. A flat board, whose upper surface forms a platform on which the planimeter can rest, is placed by the side of the cylinder so that its upper surface lies in the tangent plane. A straight-edge is fixed above the upper edge of the cylinder. The tracer of the planimeter is pressed against it and made to follow the point P on the curve. After a complete revolution of the cylinder, the planimeter will register a number proportional to the coefficient A_n or B_n.

I had an instrument of this kind made early in 1889, but it did not turn out quite as simple as its theory. It gives, of course, only one coefficient at a time, though it would not be difficult to construct it to give more terms if it were not for the mechanism required to produce the simple harmonic

motion. This always introduces a certain amount of friction if it is to work accurately. I therefore tried to do away with this, and obtained my object in the manner now to be described.

§ 4. If the definite integrals which determine the coefficients A_n and B_n be integrated by parts, we get for the former

$$n\pi A_n = \Big[y \sin n\theta\Big]_0^{2\pi} - \int_0^{2\pi} \sin n\theta \, dy,$$

the limits relating to θ.

If the original curve is continuous, the integrated part vanishes. This is not the case if there is a discontinuity, at least not if θ is retained as the independent variable.

To prove that in this case also the integrated part can be neglected, let us consider the curve in fig. 2. Let θ' be the value of θ for which the discontinuity C C′ occurs, and let y_1' be the ordinate of C, and y_2' that of C′.

The integral with regard to θ has to be broken up into two, the first going from 0 to θ', the second from θ' to 2π. The integrated part, therefore, gives

$$y_1' \sin n\theta' - y_2' \sin n\theta',$$

and this, in general, does not vanish.

The remaining integral has to be taken for the two parts of the curve from A′ to C and from C′ to B′, if the curve is not made continuous. But if the curve is made continuous, we have also to take the integral for the intervals from C to C′, and from B′ to B″. For these $d\theta$ vanishes, but not dy. This gives in addition the integrals

$$-\int_{C}^{C'} \sin n\theta \, dy = -\sin n\theta' \int_{C}^{C'} dy = -\sin n\theta' \, (y_2' - y_1');$$

hence just the terms obtained before from the integrated part.

The second integral for the interval B′ B″ vanishes because it is multiplied by $\sin 2n\pi$. In case of the coefficient B_n this is not the case, but then the integrated part also contains more terms which equal it. Hence :—

If the integration is performed with regard to y *we get*

$$n\,A_n = \frac{-1}{\pi}\int \sin n\theta\,dy, \quad n\,B_n = \frac{1}{\pi}\int \cos n\theta\,dy,$$

both taken over the whole continuous curve from A′ *to* B″. If the integration be continued from B″ to A′ on the line parallel to the axis of x nothing is added to the integral, because here dy vanishes.

For the Analysers now to be described this extension of the integration should always be made in order to eliminate certain errors of the instrument.

These new integrals are of a very different form from the old ones, and require accordingly a different mechanism. As the tracer of the instrument follows the curve, each dy has to be multiplied by $\sin n\theta$ or $\cos n\theta$. In other words, we have to decompose the dy for each element of the curve into two components at right angles to each other, of which the one makes an angle θ with the axis of x, and then add all components of each kind to get A_n and B_n.

Originally I did this by aid of a pair of registering-wheels such as are used in Amsler's well-known planimeter, the axes of the two being at right angles. If such a wheel moves along a straight line of length p, making an angle $n\theta$ with its own axis, it will register not p but $p\cos n\theta$, whilst the second wheel at right angles to it gives $p\sin n\theta$.

A model of this instrument was made in 1889.

The curve is wrapped round a horizontal cylinder. Parallel to this a carriage runs on a rail carrying the tracer which moves along the upper edge of the cylinder. It also carries a vertical spindle which has the two registering-wheels attached to it. These roll on a horizontal platform. If, now, this spindle is made to turn through an angle $n\theta$ when the cylinder has turned through an angle θ, and if the tracer is made to follow the curve, then the two registering-wheels will give the coefficients A_n and B_n. For the details of the construction I must refer to Prof. Dyck's Catalogue, p. 213.

§ 5. The next improvement is due to Mr. A. Sharp, of the Teaching Staff in the Guilds' Central Technical College. Having used my model, he brought me a design in which the principles explained were realized in a different manner.

Among the alterations introduced one struck me as being of importance. It consisted in an inversion of the motion, the curve being drawn on the drawing-board and the instrument made to move over it whilst the registering-wheels rolled on the paper.

It seemed to me that we had now all the elements needed for a really good instrument, and only wanted a practised instrument-maker to realize it. I therefore called in 1892 on Coradi in Zürich, well known for his planimeters and integrators. He set to work at once and sent me in a short time a drawing of his construction, and it is due to his skill that the instrument has, at last, reached a high degree of perfection. One Analyser has been made for the Guilds' Central Technical College, which I shall describe. But I must mention at once that Herr Coradi has since greatly improved it, so much so that it is now one of the most perfect integrators made.

§ 6. Fig. 3 shows an instrument of Coradi's second design. This will help to explain the first.

There is first of all a solid frame whose base is a long rectangle. It rests with three wheels on the drawing-board. One of these, D, in the middle of the front, serves merely as a support. The other two, E, E, are fixed to the ends of a long axle which runs along the back of the frame. This may be called the "shaft." It is placed parallel to the axis of x. The instrument can, therefore, roll over the paper in the direction of the ordinates y.

If thus moved through a distance dy, the shaft will turn through an angle proportional to dy. The shaft carries any required number of short "cylinders." In the figure there is one marked C situated in the middle of the shaft.

Above each of these cylinders is a vertical "spindle" S, whose geometrical axis cuts that of the shaft. In the new instrument each spindle carries one or two disks, H_3, H_4, in fig. 3; but in the old construction one crown wheel with its teeth pointing upwards, by aid of which the spindle is turned. At the lower end the integrating apparatus proper is attached, which is quite different in the two designs. But before explaining this let me describe how the spindle is turned.

Along the front of the frame runs a carriage W, to which the tracer F is fixed. This can be moved through a distance equal to the base c to which the curve is drawn. To the carriage a silver wire is also attached, which in the new design is stretched along the front of the frame and then by aid of guide-pulleys l, l over one of the disks H on top of the spindle S (see fig. 3). By giving the disk H a suitable diameter the spindle can be made to turn n times round, whilst the tracer describes the whole base. In the old instrument the wire only drives an extra spindle in the middle of the frame, which by aid of wheelwork drives all the working spindles. If the tracer on following the curve has reached a point P, then the spindle will have turned through an angle $n\theta$, where θ corresponds to the x of P.

If, now, the spindle had at its lower end two registering-wheels at right angles to each other rolling on the drawing-paper, we should have in principle my old model (§ 4) with Sharp's inversion. Instead of this Coradi gave each spindle one registering-wheel and made this roll on the cylinder C. This requires for each registering-wheel a separate spindle, hence two for each pair of coefficients A_n and B_n. It substitutes, however, the rolling on a smooth surface for that on the rough surface of the paper. The instrument made according to this design for the Guilds' Central Technical College has five such pairs, so that on going once over the curve the first five pairs A_n and B_n are obtained. The extra spindle which is driven by the silver wire contains, however, three extra disks, making four in all. If the wire is stretched over the top disk we get, as stated, the coefficients for $n=1$, 2, 3, 4, 5. The second pulley has half the diameter, so that the spindles turn twice as fast if the wire is stretched round it. Thus in going over the curve a second time we get the new coefficients for $n=6$, 8, 10. The remaining two disks give similarly the coefficients for $n=7$ and 9 respectively. Hence on going four times over the curve we get ten pairs of coefficients. In most cases the five pairs obtained at once will be amply sufficient.

For the details of the construction I must again refer to Prof. Dyck's Catalogue (*Nachtrag*, p. 34) and only mention a few points. The axis of a registering-wheel lies in the

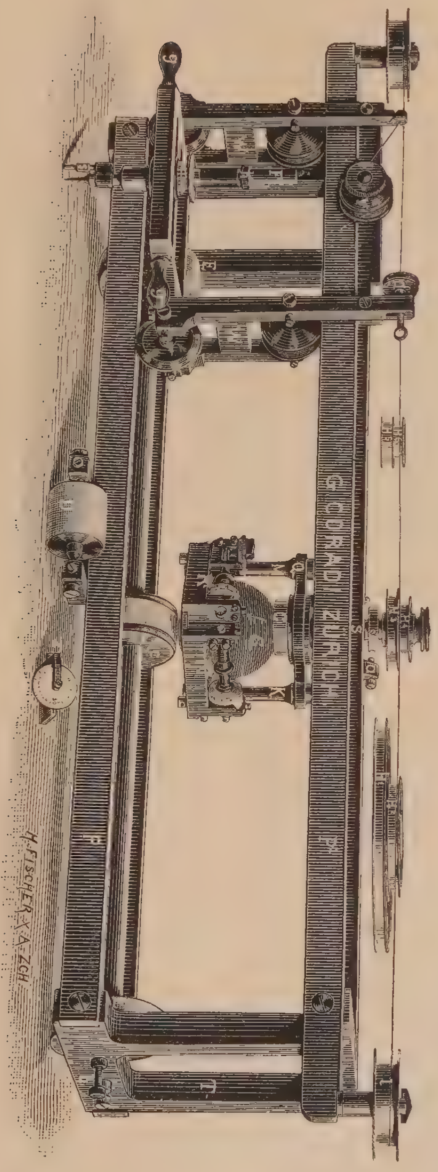


Fig. 3.

diameter of a horizontal ring which is attached to the lower end of the spindle by aid of an elastic vertical steel plate. This presses the wheel against the cylinder, securing contact. On testing the instrument it was found that this plate was liable to slight torsion which affected the readings. It showed a number of other drawbacks of more or less importance. One is that the registering-wheel not only rolls but also slips. This slipping is absent in the Analyser of Lord Kelvin, who has dwelt strongly on the importance of avoiding it.

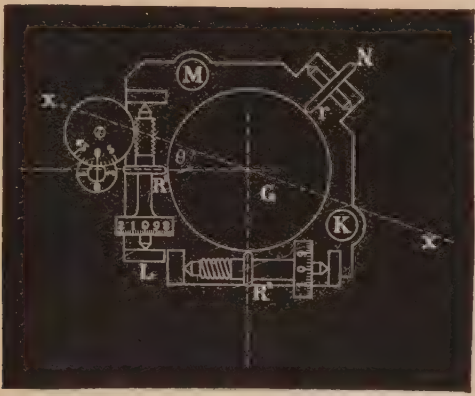
There was also a serious difficulty in taking the readings. The instrument registers up to 20 centim. If the zero-point has passed the index which gives the reading, 20 centim. have to be added or subtracted. Every one who has used a planimeter is accustomed to this, and knows how to take account of it, for he can either estimate the area sufficiently to see which correction is necessary, or he can go rapidly over he curve again, watching the zero-point. Neither method is possible with an Analyser which gives a large number of readings at once. The new instrument is therefore constructed to record up to 200 centim.

§ 7. Last summer at the Munich Exhibition Herr Coradi submitted a new arrangement to me to obviate some of the imperfections of the instrument described, and this he has since carried out with an ingenuity which I cannot enough admire. He has practically got rid of all the imperfections of the old Analyser, and has now produced an instrument which, I fancy, leaves nothing to be desired. He himself says it is the best instrument of any kind he has yet made. The chief alteration is this, that he interposes between the registering-wheel at the lower end of the spindle and the cylinder a perfectly free glass sphere.

The "spindle" has now *firmly* attached to its lower end a square frmae K L M N (comp. figs. 3 and 4) by aid of two solid rods K and M, instead of carrying the ring connected by aid of an elastic spring. This frame holds two registering-wheels R^1 and R^2, whose axes K L and L M are at right angles. Between these lies the glass sphere G, resting with its lowest point on the cylinder belonging to the spindle. A third wheel r at N is by aid of a spring pressed against the

sphere to secure contact between the latter and the registering-wheels. If, now, the tracer follows the curve this frame will turn with the spindle, the three wheels will carry the sphere with it, which will turn pivot-like on its lowest point. If, as

Fig. 4.



in fig. 4, the plane of one wheel R^1 makes with the axis of x an angle $n\theta$, and if in this position the tracer, and with it the whole instrument, is moved through the distance dy, the "shaft" will turn proportionally to dy. This will set the sphere turning about its horizontal diameter xx parallel to the shaft, and this motion will be communicated to each of the registering-wheels. It will be seen at once, if q denotes the radius of the sphere, the point of contact of the sphere and the wheel R^1 is at a distance $q \sin n\theta$ from the axis of the sphere, that therefore the turning communicated to this wheel will be proportional to $dy \sin n\theta$. Similarly the other wheel will turn proportionally to $dy \cos n\theta$. If the tracer moves through the whole curve, these two wheels will therefore register numbers proportional to A_n and B_n. The dimensions are so chosen that the readings give nA_n and nB_n in centimetres.

It will be seen that now one spindle does the work of two in the old instrument. There is, further, no slipping of any kind in the integrating apparatus.

Another improvement is that the wheelwork for turning

the spindles is done away with. Each spindle is turned directly by the silver wire, and thus any slackness in the wheels is done away with.

It has also been possible to introduce an arrangement to set all spindles to zero after the wire has been tightened.

Lastly, the readings are taken with much greater ease as the registering apparatus is well exposed to the eye.

In order that the instrument may work accurately it is necessary that the point of contact of the sphere with its cylinder should lie in the geometrical axis of the spindle. But it is practically impossible to secure this. This point will therefore describe a small circle on the cylinder and this will turn the sphere about some horizontal diameter, and therefore also the registering-wheels. It is of importance to eliminate the error thus introduced. This is done by bringing the tracer back to the starting-point A on the curve by moving it from B to A (figs. 1, 2) parallel to the axis of x. The sphere will hereby repeat the motion which produced the error, but in the opposite sense, and therefore completely cancel it.

§ 8. The first instrument of this kind has been made for Prof. Klein at Göttingen. It contains one spindle, as in fig. 3. Going once over the curves it gives therefore one pair of coefficients. To get more, disks of different diameter have to be used to drive the spindle. Of these six are provided. Since then two further instruments have been finished; one with five spindles, which goes to Moscow, the other, with three spindles, for Prof. Weber in Zürich. The experience gained in the making of the Göttingen instrument has enabled Coradi to introduce a number of small improvements, with the result that the carriage runs in the Moscow instrument, where it has to drive five spindles, as easily as in the one for Göttingen with only one spindle.

He has also introduced a celluloid ring below the sphere, which on being raised presses the sphere against a similar ring above, thus preventing any damage to the integrating apparatus when the instrument is not being used.

Note.—At the request of Herr Coradi I add the statement that the idea of the new integrating apparatus, consisting of a

sphere with two recording-wheels at right angles to each other, is not his own, but is due to Herr Max Küntzel, of Charlottenhof, near Königshütte in Silesia. Herr Küntzel invented the arrangement for an instrument designed to determine the coordinates of the vertices of a polygon, and submitted his design to Herr Coradi for the construction of such an instrument.

X. *Harmonic Analyser, giving Direct Readings of the Amplitude and Epoch of the various constituent Simple Harmonic Terms.* *By* ARCHIBALD SHARP, *B.Sc., Wh.Sc., A.M.I.C.E.**

LET the curve (fig. 1) be that represented by the equation $y=f(x)$, the scale of abscissæ being such that the period is 2π. Suppose a wheel W to roll on the paper (fig. 2),

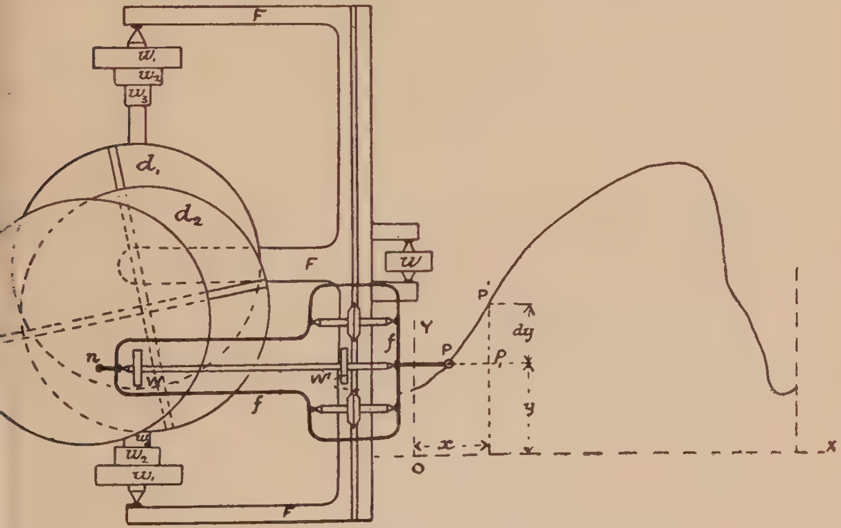


FIG. 1

and to be connected with a tracing-point P (fig. 1) in such a manner that as P moves uniformly in the direction

* Read April 13, 1894.

OX the axis of the wheel W turns uniformly in a horizontal plane, and the distance rolled through by the wheel during any short interval is equal to the corresponding displacement of the tracer P in the direction OY. If the axis of the rolling wheel W makes one complete turn while the tracing-point P moves over one complete period of the curve (fig. 1), the point of contact of the rolling wheel will describe a curve $Opp'R$ (fig. 2).

Let OY′ (fig. 2) be the initial direction of the plane of the

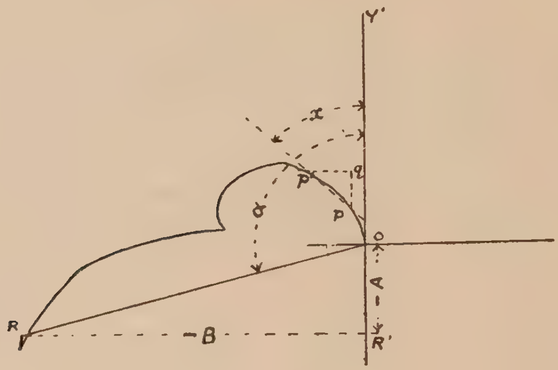


FIG. 2

rolling wheel, *i. e.* corresponding to zero abscissa of the tracer (fig. 1). Let P be any point on the curve fig. 1, p the corresponding point on the curve fig. 2. Let P′ and p' be two corresponding points infinitely close to P and p respectively. In fig. 1 draw $P'P_1$ and PP_1 parallel to OY and OX respectively, and in fig. 2 draw $p'q$ and pq respectively at right angles and parallel to OY′.

Then

$$pp' = P_1P' = dy,$$

$$p'q = pp' \sin x = \sin x\,dy,$$

$$pq = pp' \cos x = \cos x\,dy.$$

Draw RR′ perpendicular to OY′ (fig. 2), R being the position of p corresponding to $x = 2\pi$. Then

$$RR' = \Sigma p'q = \int_0^{2\pi} \sin x\, dy,$$

$$OR' = -\Sigma pq = -\int \cos x\, dy.$$

In the Fourier expansion

$$\left.\begin{aligned} y=f(x) = A_0 + A_1 \sin x + A_2 \sin 2x + \,.\,.\, + A_n \sin nx + \\ + B_1 \cos x + B_2 \cos 2x + \,.\,.\, + B_n \cos nx + .\,, \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} A_1 &= \frac{1}{\pi}\int_0^{2\pi} y \sin x\, dx = \frac{1}{\pi}\int \cos x\, dy = -\frac{1}{\pi} OR' \\ B_1 &= \frac{1}{\pi}\int_0^{2\pi} y \cos x\, dx = \frac{1}{\pi}\int \sin x\, dy = -\frac{1}{\pi} RR' \end{aligned}\right\} \quad . \quad . \quad (2)$$

Also $f(x)$ may be expanded in the form

$$A_0 + C_1 \sin (x - \alpha_1) + C_2 \sin (2x - \alpha_2) + \,.\,.\, C_n \sin (nx - \alpha_n), \quad (3)$$

A_n, B_n, C_n, and α_n being connected by the relations

$$\left.\begin{aligned} A_n &= C_n \cos \alpha_n \\ -B_n &= C_n \sin \alpha_n \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (4a)$$

or

$$\left.\begin{aligned} C_n^2 &= A_n^2 + B_n^2 \\ \tan \alpha &= -\frac{B}{A} \end{aligned}\right\} . \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

From (2) and (4*b*) it is evident that OR (fig. 2) is equal to πC_1, and the angle Y′OR is equal to α_1.

If now the axis of the rolling wheel W makes n turns while the tracer P moves over one complete period of the curve (fig. 1), the corresponding values of OR and the angle Y′OR will be $n\pi C_n$ and α_n respectively.

Various arrangements of mechanism are suggested for connecting the rolling wheel with the tracer so as to satisfy the above conditions; the following seems the most suitable, as it can be adapted for an instrument to give more than one simple harmonic constituent term for one tracing of the curve.

The motion of the rolling wheel relative to the paper is compounded of two simple movements :—(*a*) a pure rolling,

the distance rolled being equal to dy the element described by the tracer P; (b) a motion of rotation, the point of contact of the wheel with the paper being the centre of rotation, and the angle turned through from the initial line being proportional to x the abscissa of P. The relative motion will, therefore, be the same if the wheel be rolled along a straight line *fixed* in the instrument, while the *paper* is made to turn, the centre of rotation of the paper being the point of contact of the wheel with it which is continually varying in position. Fig. 1 represents diagrammatically the mechanism. The curve to be analysed is drawn on a flat sheet of paper and placed on a drawing-board. The carriage FF, which forms the base of the instrument, is supported by an axle with two equal wheels w_1 and a third wheel w which roll on the paper, the direction of motion of the carriage being OX. A disk d_1 mounted on a vertical spindle is driven by a pair of bevel wheels by the axle w_1 w_1. A long key on the upper surface of this disk fits into a groove on the under surface of a disk d_2, which is thus free to move in a straight line relative to disk d_1. A groove on the upper surface of disk d_2 at right angles to that on its lower surface has a key from the lower surface of disk d_3 resting in it. Thus the disk d_3 always turns with disk d_1, although any point on disk d_3 may be made the centre of rotation ; the three disks being kinematically equivalent to Oldham's coupling for the transmission of motion between two parallel shafts. The keys and grooves would be replaced, in an actual instrument, by wheels and rails, in order to diminish frictional resistance.

The tracing-point P is mounted on a smaller carriage f, which is free to run in the direction OY relative to the main carriage F. This smaller carriage carries also the rolling wheel W which rolls on the disk d_3. The rolling wheel W should be spherical in form, and of as small diameter as possible, so that its surface of contact with the paper on disk d_3 approximates to a point. The friction between wheel W and disk d_3 is great enough to prevent any relative sliding. As the tracer P moves over the curve (fig. 1) the point of the wheel W will describe on the disk d_3 the curve Opp'R (fig. 2). To ensure that, as the tracer P is moved in the direction OY, the wheel W will *roll* on the disk d_3 the same

distance and not displace it relative to disk d_1, a wheel W′ of the same diameter as W is mounted on the same spindle and rolls on a fixed portion of the carriage FF. If W′ be compelled to roll, W must roll on the disk d_3 an equal amount.

The actual shape of the curve Opp'R (fig. 2) is of no importance, the initial and final points being all that are required. A needle or pencil n may therefore be carried at any convenient part of the carriage f, and the initial and final positions O and R marked by it. The direction of the initial line OR will be recorded on the disk d_3 by making two marks with the needle n as the tracer P moves along the line OY (fig. 1).

The gearing must be such that the disk d_3 turns once while the tracer P describes one complete period of the curve. If now pairs of equal wheels w_2 w_2, w_3 w_3, . . . of diameters $\frac{1}{2}$, $\frac{1}{3}$, . . . of w_1, be made to roll on flat rails lying on the paper, the values of c_2, α_2, c_3, α_3, . . . are obtained in succession, one pair of coefficients for each tracing of the curve.

This instrument has the advantage over any Harmonic Analyser previously designed that it gives directly the quantities—amplitude and epoch—of each simple harmonic term which are required; all other instruments, as far as I am aware, giving the coefficients A_n and B_n, from which C_n and α_n are calculated.

It is remarkable that no adjustments have to be made before using the instrument, the initial position of the disk d_3 having no influence on the curve Opp'R described on it. There is no part of the instrument which demands *excessive* accuracy of construction. The accuracy and delicacy of the instrument depends on the accuracy with which the line OR and angle Y′OR can be measured, and will be quite as great as that with which the original curve fig. 1 is drawn.

In some cases there will be a danger that the disk d_3 may not be large enough to contain the complete curve Opp'R (fig. 2). If the rolling wheel W is about to roll off the disk, a mark should be made with the needle n, and keeping the tracer P in the same position, the disk d_3 should be moved by hand into any other convenient position relative to disk d_1, a new mark made with the needle, and the movement of the

tracer P may then be proceeded with. The final line OR can then be easily built up from its separate parts.

Since writing the above I have designed an inversion of the mechanism described above giving a simple compact instrument, which I may have the pleasure of describing later on.

Discussion.

The President said a description of Mr. Sharp's Harmonic Analyser, giving direct readings of the amplitude and epoch of the various constituent simple harmonic terms, had been sent in.

This machine requires no adjustments to be made before using. The amplitude is given by the length of a line joining the initial and final portions of the point of contact of a roller with a rotating disc, whilst the epoch is determined by the angle which this line makes with the plane of the roller in its initial position.

Prof. Perry congratulated Prof. Henrici on the success attained with his analysers. Referring to Planimeters, he said the average error made in working out indicator diagrams with Hine and Robertson's instrument was only about one-third that made with Amsler's. After pointing out the great importance of Fourier's Series to practical men, and especially to electrical engineers, he said that in studying reciprocating motions, such as those of pistons, valve-gears, &c., it was most useful to resolve the motion into its fundamental harmonic motions and its overtones. In this way remarkable differences could be seen between various motions which have the same fundamental, and which are usually considered equivalent. In the 'Electrician' of Feb. 5th, 1892, he had published the numerical work for a given periodic curve developed in Fourier's Series, and he now exhibited a graphical solution done by one of his students, who was probably the first to carry out the late Prof. Clifford's idea of wrapping the curve round a cylinder and projecting it on different planes. Prof. Henrici had, he said, based the construction of his first analyser on Clifford's method, but used the Henrici principle (viz.,

$$\int y \sin \theta \, dy = \int \cos \theta \, dy,$$

when integrated over a complete period) to explain the later machines. As a matter of fact the first machine, in which the coefficients were determined by the Amsler planimeter carried by a reciprocating tangent plane, was a beautiful example of the Henrici principle, and he, Prof. Perry, saw great possibilities before it. The defects in the first instrument were mechanical, and could be got over by increasing the amplitude of the harmonic motion. Not only was the machine useful for Fourier expansions, but by giving suitable motions to the tangent plane developments of arbitrary functions in spherical harmonics, Bessel's functions, Lame's functions, and other normal forms could be determined. He had designed a machine which, on Prof. Henrici's principle, develops arbitrary functions in Bessels, and hoped to have shown it in working order at the meeting; the Easter holidays had prevented its being finished in time. In this machine the motion is given to the table by a cam and roller, the cam being shaped so that the displacement of the table is $x \times J(x)$ when the shaft turns through an angle proportional to x. The revolving cylinder is driven by variable gearing from the cam-shaft. By using curves of other shapes, developments in many normal forms may be obtained; the machine is therefore of general analytical use. An example of development in Bessels worked out arithmetically by two of his students, Messrs. Hunt and Fennell, was given, and the process of performing the integrations by the machine described.

Prof. Boys, speaking of arithmometers, said Prof. Selling's machine had several inconveniences. In the first place, it occupied a large space, and the projecting racks were apt to upset things just behind the machine. Secondly, the result of any operation was indicated by continuous motion, and therefore cannot be read off instantly with certainty. On the other hand, the "Brunsviga" machine was very compact and convenient, the only serious defect being that one cannot carry on figures obtained as the result of one operation to work with again, as was possible in the well-known Colmer machine. As another improvement, he suggested that the two sets of numbers on the wheels showing the result of any operation should be coloured differently, so that it would be

easy to see whether multiplication or division had been performed. The labour of operating with large digits could then be considerably reduced with certainty. For example, in multiplying by 2998, instead of 28 $(2+9+9+8)$ turns of the handle, 5 would be sufficient, viz., three in the forward direction and two backward, thus giving $30\overline{02}$. In his opinion logarithm tables were not nearly so convenient for ordinary calculations as this machine.

Mr. A. P. TROTTER described how by the use of templates cut to suitable shapes one could obtain true curves from those given by recording voltmeters and similar apparatus.

Mr. YULE said he had recently seen the newest analyser made by Coradi for Prof. Weber, and was present when it was tested by the latter on a simple harmonic curve. It gave excellent results, the errors not amounting to 1 part in 2000. Speaking of the "hatchet" planimeter, he thought that the first one was exhibited by Mr. Goodman at the Institution of Civil Engineers.

Mr. A. SHARP remarked that since last meeting he had designed an inversion of the mechanism in his harmonic analyser, which made it much more practical.

Prof. HENRICI, in reply, said the uses of his first machine suggested by Prof. Perry might lead to great developments in this subject. Lord Kelvin had shown that with the sphere and roller integrator, products of two functions, such as $f(x)\,F(x)dx$ could be got. Referring to Prof. Boys's criticism on the Selling arithmometer, he did not consider the difficulty in reading off the result at all serious. Mr. Trotter's method of solving problems by templates might be very useful. Speaking of the "hatchet" planimeter, he said he believed it was first brought out in Denmark; Mr. F. W. Hill, of the City of London School, had sent him a solution of its action. Mr. Sharp, he said, had made a very considerable improvement in his machine, and the elements of this integrator may be useful for other purposes.

XI. *Remarks on* Prof. Henrici's *Paper made by* Prof. Perry, *F.R.S., in which he describes a Simple Machine which may be used to develop any Arbitrary Function in Series of Functions of any Normal Forms**.

I congratulate Prof. Henrici, first upon his success in these Analysers, with which I shall presently form a practical acquaintance when the latest of them yet constructed reaches me from Zurich, second on the admirably clear way in which he described them to us.

I have had no experience with the hatchet, that simplest of all planimeters; but with regard to the Robertson-Hyne instrument, which comes to us from America, and which in the size here exhibited is well suited to Indicator-diagram work, I can say that some of my students instituted a careful comparison between it and the Amsler which they use for Indicator diagrams, and they found that the average error with it was about one third of that with the Amsler.

We know that in mathematical physics generally the development of an arbitrary function in a Fourier's Series is often of great importance; but I wish to say that this subject is becoming of greater and greater importance to the practical man—the engineer.

Thus in alternating-electric-current work, all the disturbing, distracting, dangerous troubles are considerably increased when the currents are not simple harmonic functions of the time. With two-phase or three-phase currents, if the amplitudes are not equal, the rotating magnetic field neither remains of constant strength nor has it constant angular velocity; and if there are overtones we have extra fields of changing magnitude, which rotate irregularly at two or more times the speed of the fundamental.

I have long thought that mechanical engineers need such instruments as Prof. Henrici has designed, if only to familiarize them with the ideas of Fourier. It has for some time been my habit, when studying with students any kind of reciprocating motion of a piece of machinery, to resolve the motion into its fundamental harmonic motion and overtones.

* Read April 13, 1894.

For example, if one is studying the forces causing the motion, one ought to keep in mind that the reciprocating motion which (speaking rather vaguely) requires the smallest forces or moments of forces to produce it, is the simple harmonic motion. The accelerating forces due to an octave are four times as great as for a fundamental of the same amplitude. The motion of the piston of a steam-engine is, with sufficient exactness for practical calculations, a fundamental of amplitude r and an octave of amplitude $\frac{r^2}{4l}$, where r is the length of the crank and l the length of connecting-rod.

A special graphical method of study may be discovered and employed for any special motion; but for applicability to reciprocating motions in general I know of nothing to compare with the method of study which is based on finding the fundamental motion and one or more overtones.

Again, the difference between one kind of slide-valve motion and another may be exceedingly great, practically, and yet the theories found in books show no difference at all. Indeed, the complete mathematical methods of study are too troublesome, but the mathematics of link motions and radial valve-gears become very simple when we consider, not merely the fundamental simple harmonic motion, which is all that is usually studied, but the octave, which is found to help or hurt in the various forms.

I was first attracted to this subject when studying the beautiful but little-known valve-motion invented long ago by Sir F. Bramwell, in which the only overtone is three times the fundamental.

Given any function completely, we can by a numerical method, and with as much accuracy as we please, develop it in Fourier's Series. In the 'Electrician' of Feb. 5th, 1892, I published the numerical work of one example calculating from 23 ordinates. In the sheet which I here exhibit one of my students, Mr. Fox, has done the same work by a graphical method. Probably he is the very first to carry out the idea of the late Prof. Clifford by descriptive geometry*. That is,

* *Note added May 29th.*—The descriptive geometry method is fairly quick, and may be made as accurate as one pleases, but of course it cannot compare in quickness with the Henrici Analyser.

It is obvious that by properly shaping one's cylinder, wrapping the

we have imagined the curve to be wrapped round the cylinder, and it was surprising to find how rapidly its projections could be drawn upon the two planes and their areas obtained by the planimeter. We then imagined the curve to be wrapped twice round and the projections drawn and their areas taken. I wish I had time to dwell upon the interesting problems that arose during the work, for example as to whether the area was to be taken as positive or negative. However many loops such a figure may possess, the well-known rule for autotomic plane circuits (Thomson and Tait's 'Elements,' §445) is really attended to by the planimeter. The direction of motion of the tracer must be that in which x increases on the real curve. I here give the results :—

The values of the arbitrary function to be analysed were really calculated from

$$y=10+5\sin\left(\frac{2\pi}{c}x+30^\circ\right)-\sin\left(\frac{4\pi}{c}x-60^\circ\right).$$

The result obtained numerically and published in the 'Electrician,' using 23 ordinates, was

$$y=9{\cdot}966+5{\cdot}039\sin\left(\frac{2\pi}{c}x+29^\circ{\cdot}9\right)-1{\cdot}053\sin\left(\frac{4\pi}{c}x-55^\circ{\cdot}3\right).$$

The result now obtained graphically is

$$y=10{\cdot}01+5{\cdot}0096\sin\left(\frac{2\pi}{c}x+30^\circ{\cdot}38\right)-1{\cdot}0099\sin\left(\frac{4\pi}{c}x-59^\circ{\cdot}22\right).$$

---

curve round it, and then finding the area of it, projected on a plane parallel to the axis, one may develop an arbitrary function in a series of any normal forms. Thus if $Q(x)$ is any tabulated function of x, and y is the arbitrary function of x, and we wish to find the integral $\int_0^a y \,.\, Q(x) \,.\, dx$, the shape of the curve which must be used instead of a circle in the Clifford construction is easy to find. It must be such that the cosine of the angle which the short length δx of the curve makes with the trace of the plane on which the projection is to take place shall be proportional to $Q(x)$, and several easy methods of drawing the curve or a series of such curves may be found. Once found, there is no more difficulty in developing any new arbitrary function in any series of normal forms than Mr. Fox found with his Fourier Series. A series of curves will be needed for a development in Zonal Harmonics, but only one curve will be needed for the Zeroth Bessels. These curves, or shapes of sections of cylinders, I am now proceeding to draw on a sufficiently large scale for exact work.

It is curious that Prof. Henrici should have based the construction of his first or 1889 instrument on the beautiful idea of the late Prof. Clifford, and not on what I call the Henrici principle. He gives the Henrici principle to explain the later instruments, and does not seem to see that his first instrument is the most beautiful example of its application. I take the Henrici principle to be that

$$\int y \,.\, \sin\theta \,.\, d\theta = \int \cos\theta \,.\, dy,$$

the integrations being for a whole period. Well, in his first instrument, whilst its tracer moves through the distance dy, the ordinarily fixed part of the planimeter now has a displacement $\cos\theta$, and this is the same as if in the ordinary use of the instrument a curve is being traced whose ordinate is $\cos\theta$.

It is only on the assumption that the Henrici principle applies to his first instrument, that I venture to say that the following analyser is on the Henrici principle. We have at present to develop functions in sines and cosines, spherical harmonics and Bessel functions, because we know that when we have effected such developments we can convert them at once into the solutions of certain physical problems. As time goes on we shall require developments in many other normal forms. I propose to describe a machine which will effect any such development. I mean, that my machine will evaluate the integral $\int_0^a f(x) \,.\, Q(x) \,.\, dx$, where $y = f(x)$ is an arbitrary function of x and $Q(x)$ is any tabulated function. Following Henrici, we convert the required integral into

$$\Big[f(x) \,.\, H(x) \Big]_0^a - \int H(x) \,.\, dy,$$

where $H(x)$ is the integral of $Q(x)$ and may be tabulated as $Q(x)$ is tabulated. Now in many cases the part between the square brackets is zero, but this is of little consequence in comparison with the fact that the part $\int H(x) \,.\, dy$ may be evaluated by a machine somewhat like Prof. Henrici's first or 1889 instrument. I have worked with this 1889 instrument, and I am not disposed to think it so inaccurate as its inventor thinks it. Its defects are really defects of mechanical construction; for example, the amplitude of the simple harmonic motion of its table is very much too small.

I have already put my machine in hand and hoped to exhibit it here to-day in working order, but unfortunately the Easter holidays have prevented it being finished in time. It is arranged to develop an arbitrary function in Bessels of the zeroth order, or rather Fourier cylindric functions. Thus it is required to determine the constants A_1, A_2, &c., in

$$f(r) = A_1 J_0(\mu_1 r) + A_2 J_0(\mu_2 r) + \text{\&c.}, \quad . \quad . \quad . \quad . \quad (1)$$

where μ_1, μ_2, &c. are the successive roots of some such equation as

$$J_0(\mu a) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or

$$\mu a J_1(\mu a) - \lambda J_0(\mu a) = 0, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where λ has a given value.

It is well known that

$$A_s = M \int_0^a r f(r) \,.\, J_0(\mu_s r) \,.\, dr,$$

where $M = 2/a^2 \, [J_1(\mu_s a)]^2$, if μ_1, μ_2, &c. are the roots of (2), and

$M = 2\mu^2_s/(\lambda^2 + \mu^2_s a^2)\,[J_0(\mu_s a)]^2$, if μ_1, μ_2, &c. are the roots of (3).

In every case the practical difficulty consists in finding the integral. I exhibit to the Society an easy example of such an analysis worked out numerically (I suppose that such a thing has never been done before) by two of my students, Mr. H. F. Hunt and Mr. W. Fennell.

It will be seen that the work is rather tedious. It was made more tedious by their having found it necessary to calculate numbers and tabulate them in a handy form, interpolating between the numbers given in Lommel by the use of his formula. Before this work was finished we discovered Dr. Meissel's elaborate tables, from which the remainder of our handy four-figure tables is merely copied. These handy tables of $J_0(x)$ and $J_1(x)$ are at the service of the Society; they would occupy just four pages of the Journal. We have found them of practical value, but I do not know whether they are of such general value that they ought to be printed.

It is well known that $\int_0^x x J_0(x) \,.\, dx = x J_1(x)$, and hence if

we write $\phi(r)$ for $\mu r J_1(\mu r)$, and y for our arbitrary function $f(r)$, the required integral $\int_0^a rf(r) \,.\, J_0(\mu r) \,.\, dr$ is

$$\frac{1}{\mu^2}\int_0^a y \,.\, \frac{d\phi(r)}{dr}\, dr = \left[{}_0^a \frac{1}{\mu^2} f(r)\phi(r)\right] - \frac{1}{\mu^2}\int \phi(r) \,.\, dy.$$

The part between the square brackets is usually 0, but however that may be, we see that we can evaluate the integral by the machine. A curve is drawn representing $f(r)$ from $r=0$ to $r=a$ on a sheet of paper which is wrapped round a roller. a need not be equal to the whole circumference of the roller and the scale of r is unimportant. Of course the ordinate y or $f(r)$ lies parallel to the axis of the roller. It is the measurement of y in inches which my instrument analyses, as my planimeter is graduated in square inches. A table whose upper surface is in a plane tangential to the roller carries the usually fixed part and rolling wheel of an Amsler planimeter. On turning through an angle θ a handle which drives a shaft on which a properly shaped cam is keyed, the table is displaced in its own plane towards the roller, through the distance $x\,J_1(x)$, θ being proportional to x, and at the same time the roller is driven so that the paper moves circumferentially through a distance proportional to x. For this particular kind of problem a few different but definite trains of gearing might be used to connect the handle and the roller, but for general purposes I would prefer variable friction gearing to give any relative speeds that may be necessary. In my model now being constructed I am using two disks, one of which rests on the other at a point which may be altered, radially. As in Prof. Henrici's instrument, the tracing-point of the planimeter is held against a straight edge, so that it can only move along the tangent-line of roller and table whilst following the curve on the roller.

If μ is a root of $J_0(\mu a)=0$ as in the well-known drum-head problem, a being the radius of the drum-head,—in the first operation to find A_1, the gearing must be adjusted so that when the whole curve on the roller passes under the tracing-point of the planimeter, a graduated circle on the

shaft turned by the handle indicates that it has turned through an angle proportional to 2·405, which is the first value of x which satisfies $J_0(x)=0$.

It is advisable to have a pointer and a scale to indicate exactly the displacement of the table, so as to test the accuracy with which the cam performs its duties. Of course, when the graduated-circle indication is 2·405, the displacement of the table is to be $2\cdot405\ J_1(2\cdot405)$ or $-1\cdot249$ inches. The area recorded on the planimeter in square inches must now be multiplied by $2(2\cdot405)^2/a^4[J_1(2\cdot405)]^2$, and the answer is A_1.

To find A_2: change the gearing so that when the whole roller-curve passes under the tracing-point of the planimeter, the graduated circle indicates 5·5201, and check the error of the cam by noting that the displacement indication ought now to be $5\cdot5201\ J_1(5\cdot5201)$ or 1·878 inches. The area recorded by the planimeter in square inches must now be multiplied by $2(5\cdot5201)^2/a^4[J_1(5\cdot5201)]^2$. If variable frictional gearing is used, it is important that the roller should be placed on roller bearings of small resistance.

To develop an arbitrary function in Bessels of any other order, or in Fourier's Series, or in zonal harmonics, or in series of functions of any other normal forms, we have only to replace the cam by one of another shape; so that this one simple machine is suited to quite general analytical use.

XII. *On the Mechanism of Electrical Conduction.*—Part I. *Conduction in Metals.* *By* Charles V. Burton, *D.Sc.**

1. The view of electrical conduction which it is here my object to explain receives general support from more than one consideration; for it leads to the conclusion that deviations from Ohm's Law must be quite inappreciable in the case of metallic conductors, and it goes far to explain, I think, why metals are so much less opaque than their ordinary conductivities would lead us to infer. But it is not

* Read April 27, 1894.

alone on such considerations that we have to rely, for, as it seems to me, the main conclusions are capable of exact demonstration ; and accordingly it would appear most convenient to commence with a few simple theorems, seeking afterwards to account for known phenomena by means of our definite results.

2. Theorem I.

In a region containing matter, there may be (and probably always are) some parts which are perfect insulators and some parts which are perfect conductors ; but there can be no parts whose conductivity is finite—unless every finitely conductive portion is enclosed by a perfectly conductive envelope.

Before proceeding to the proof of this theorem, it may be remarked that the presence of the last clause in no way modifies any application of our result, since the space within a perfectly conductive envelope is completely shielded from the influence of external charges, currents, or magnets. In the present state of science, indeed, such words appear necessary to the completeness of demonstration, but they do not need to be considered in any of our deductions from the theorem, and for my own part I am persuaded that in reality there is nothing corresponding to the possibility which they suggest.

Consider now the case of any body whatever, at any temperature other than absolute zero. We know that electromagnetic radiations will spread out into the ether surrounding the body, and we must suppose that the intermolecular spaces within the body are also traversed by electromagnetic disturbances. Let us suppose then, for a moment, that in the molecules of the body there are some finitely conductive portions which are *not* enclosed in perfectly conductive envelopes. The electromagnetic disturbances will give rise to currents of conduction in these portions, and accordingly energy will be degraded into a form which is *not heat*, since it consists, not in the motion or relative positions of molecules or appreciable parts of molecules, or in electromagnetic disturbances of the intervening ether, but in something *much more fine-grained.* We shall thus have *a continual degradation*

of heat into energy of a lower form; for the electromagnetic "damping" of the finitely conductive bodies involves a continual drain on the energy of internal radiation, and hence indirectly on the energy of the molecules, so that heat will be automatically dissipated in the interior of the body. This process, in which the radiative molecules are continually imparting to the ether more energy than they receive in return, may be compared to the surface cooling of an isolated body which radiates towards colder surroundings.

Even if we suppose the finitely conductive bodies to be extremely small and their conductivity to be either extremely small or extremely great, it is not hard to see that the rate of absorption of heat must be tremendous; and when we consider (for example) the effect which even a very slow absorption, continued for millions of years, would have had on the temperature of our planet, we must admit that the absence of that dissipation of heat implied in the denial of Theorem I. has been established with an exactitude almost unparalleled. Thus the theorem is established.

3. In connexion with this result we are reminded that Poisson's theory of dielectrics requires the molecules of insulating substances to possess some conductive portions, though whether the conductivity of such portions is finite or infinite is of no moment in electrostatics. On the other hand, both Ampère's theory of magnetism and Weber's theory of diamagnetism suppose the existence of *perfectly conductive particles*, and are thus strongly supported by our result.

In discussing Weber's theory of diamagnetism, Maxwell* points out that the currents excited in a perfectly conductive body by any external cause are entirely confined to the surface of the body. Thus the *perfectly conductive bodies* in Theorem I. may be replaced by *perfectly conductive surfaces*, without altering any of our conclusions; but it would be hard to decide whether a perfectly conductive geometrical surface is or is not a physical possibility without knowing more of electromagnetism—not to speak of ordinary matter.

* 'Electricity and Magnetism,' 2nd ed. vol. ii. § 840.

4. Theorem II.

In metals, and in other non-electrolytes whose conductivity is finite, the transmission of currents must be effected by the intermittent contact of perfectly conductive particles.

For if there were not these intermittent contacts, any given two of the conductive particles would be either permanently in contact with one another, or permanently out of contact, and there would be only two cases to consider. If throughout the substance there extended continuous chains of (perfectly) conductive particles in contact with one another, the substance as a whole would be a perfect conductor; while in the absence of such chains of particles, the substance would be a perfect non-conductor. Finite conductivity can only exist when the contacts are intermittent.

5. An immediate corollary is

Theorem III.

If we suppose that in a substance at the absolute zero of temperature there is no relative motion amongst the molecules or amongst their appreciable parts, it follows that every substance at this temperature must have either infinite specific resistance (which need not imply infinite dielectric strength), or infinite conductivity.

For the denial of relative motion involves the denial of that intermittence of contact which in Theorem II. was shown to be necessary to finite conductivity.

This conclusion is in accordance with the experiments of Dewar and Fleming * on the resistance of pure *unalloyed* metals at very low temperatures. In the case of all the pure metals examined by these authors (platinum, gold, palladium, silver, copper, aluminium, iron, nickel, tin, magnesium, zinc, cadmium, lead, and thallium), the temperature-resistance curves are almost straight lines, and these, being produced, would pass very nearly through the point whose coordinates are zero temperature and zero resistance.

The same was *not* found to hold good for the temperature-

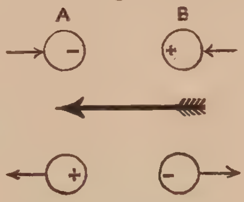
* Phil. Mag. Sept. 1893, p. 271.

resistance curves for alloys; but if these curves could be pursued far enough by experiment, they must be found, I think, to terminate at the origin of coordinates, like those of the pure metals.

6. Dissipation of Energy in a Conductor conveying a Current.

In fig. 1 let A and B be two perfectly conductive particles (whether molecules or parts of the same or of different molecules we need not consider), and let them be approaching one another. Suppose also that there is an applied E.M.F. acting from right to left (as indicated by the large arrow). Then, generally speaking, A will be negatively electrified, owing to a previous encounter with some particle farther to the left, and for a similar reason B will in general be positively electrified. When A and B collide, the usual effect is to leave A on the whole positively electrified, and B negatively electrified.

Fig. 1.



Remembering that the conductivity of A and B is perfect, let us consider what transformations of energy are effected by movements and collisions of this kind. Before the collision, A being negatively electrified is urged towards the right by the applied E.M.F., while B being positively electrified is urged towards the left: that is, A and B are urged *together*, and are gaining kinetic energy at the expense of the source of applied E.M.F. After the collision, the electrifications are, generally speaking, reversed, so that A and B are now being urged apart by the applied E.M.F., and continue to gain kinetic energy as before. Further, when particles such as A and B come into collision, so as to cause a readjustment of their electrifications, and also when they are in motion between two collisions, electromagnetic disturbances will be produced in the intermolecular ether; but since all the conductive particles are perfectly conductive, no electromagnetic energy can penetrate within them. Thus the energy expended by the source of E.M.F. which maintains a steady

current through a conductor is converted partly into additional energy of the molecules, and partly into electromagnetic disturbances of the intervening ether: that is, *the dissipated energy takes the form of heat*, as we know from experiment.

7. Ohm's Law.

In the case of a metal wire (especially one at a bright red heat), Ohm's Law has been verified with great exactitude, the results of the experiments designed by Maxwell and carried out by Chrystal being summed up by the latter in the following words *:—" If we have a conductor [of iron, platinum, or German silver] whose section is a square centimetre, and whose resistance for infinitely small currents is an ohm, its resistance (provided the temperature is kept the same) is not diminished by so much as the $1/10^{12}$ part when a current of a farad per second passes through it."

Now when a current is conveyed through a substance by intermittent contacts amongst a number of perfectly conductive particles, the effective conductivity depends firstly on the properties of the intermolecular medium, and secondly on the size, form, distribution, and movements of the particles themselves. In order that the resistance of the conductor may be sensibly constant—in order, that is, that the current transmitted may be sensibly proportional to the impressed E.M.F.—two conditions must evidently be satisfied:—

(i.) For such values of impressed electromotive intensity as exist in the intermolecular spaces (say about ·003 volt per cm.) the relation between electromotive intensity and electric displacement must be sensibly linear.

(ii.) The forces which the particles of the substance experience owing to the impressed E.M.F. must be very small in comparison with the ordinary intermolecular forces, so that during the time of a single molecular excursion the motion of no particle is appreciably influenced by the presence of the E.M.F. If we suppose that in the conducting substance we can maintain a steady distribution of temperature which is independent of the current flowing through, this second condition implies that the particles of the substance under the

* B. A. Report, 1876, p. 61 of Reports

steady distribution may be regarded as a system of perfect conductors, whose coordinates are explicitly given functions of the time, and are sensibly unalterable by an E.M.F. impressed upon the system from without. This condition, combined with (i.), will evidently give us Ohm's Law.

Now the forces actually present and tending to modify the heat-movements are of two kinds: electromagnetic and electrostatic.

(a) *Electromagnetic Forces.*—The passage of a current through a conductor gives rise to a magnetic field, which may or may not appreciably affect the conductivity. The thin iron wire used by Prof. Chrystal was ·0021 cm. in radius, and the greatest value of the magnetic force due to a current of 1 ampere per square centimetre of cross section would be in absolute measure about ·0013 (at the surface of the wire), the square of the greatest magnetic force being thus about ·0000017. The average value of (magnetic force)^2 over the cross section of the wire would be half of this, or ·00000085; that is, about ·0000039 of the square of the terrestrial "total force" in these parts.

Now Lord Kelvin found * that the change of resistance due to *transverse* magnetization of an iron plate by a powerful Ruhmkorff electromagnet was only just decided enough to be distinctly appreciated with the apparatus which he employed, and we may therefore conclude that in Prof. Chrystal's iron wire no perceptible change of resistance could have been produced by the magnetic field of the current. In other metals the effect must be still more insignificant.

On the other hand, the *longitudinal* magnetization of an iron wire perceptibly increases its electrical resistance, so that it would be easy to construct a simple conductor whose resistance at a given temperature was a function of the current-strength. For let a flat bobbin be wound with iron wire, so that each turn has the form of an elongated rectangle, and then let a further quantity of iron wire be wound in a similar circuit embracing the first. Finally let the coils be joined in series with a source of E.M.F. When a current is sent through the circuit, each coil will magnetize longi-

* Phil. Trans. 1856, especially pp. 747–749.

tudinally some parts of the wire of the other coil, and so, for a given temperature of the wire, the resistance will increase with the current.

(b) *Electrostatic Forces.*—Let us attempt to calculate the electrostatic energy per cubic centimetre which a mass of iron possesses in virtue of a current flowing through it with a "density" of 1 ampere per cm.^{2} To do this we must assume some value for the specific inductive capacity of iron*, and in order to take a sufficiently unfavourable view of the question, let us assume the value to be as high as 20. Taking the specific resistance of iron in electromagnetic measure to be 10,000, and remembering that 1 ampere $=\cdot1$ absolute unit, we have for the electromotive intensity 1000 electromagnetic units of potential per cm., *i.e.* $1000\div(3\times10^{10})$ electrostatic units per cm. Hence the electrostatic energy per c. c. due to the impressed E.M.F.

$$=\frac{20}{8\pi(3\times10^{7})^{2}}\text{ ergs};$$

while to calculate the *thermal* energy per c. c. at "a bright red heat"—the temperature of the iron in the British Association experiments—we have:—

Temperature above absolute zero (say)	$=727+273$
	$=1000$ Cent. degrees,
Density of iron	$=7\cdot8$,
Specific heat	$=\cdot113$,
One gram-water-degree of heat .	$=42\times10^{6}$ ergs.

Thus (roughly speaking) the thermal energy per c. c. reckoned from absolute zero

$$=1000\times7\cdot8\times\cdot113\times42\times10^{6}\text{ ergs}.$$

A comparison of these results gives

$$\frac{\text{electrostatic energy due to impressed E.M.F.}}{\text{thermal energy}}=\frac{1}{4\times10^{22}}$$

* In electrostatic measurements conductors appear to have an infinite specific inductive capacity; but here, where the potential really varies from point to point through the metal, it is the true (finite) specific inductive capacity which concerns us.

only, even on our assumption that the specific inductive capacity of iron in electrostatic measure is as high as 20. If we suppose that half the thermal energy is potential and half kinetic, then the electrostatic energy would be $1 \div (2 \times 10^{22})$ of the thermal kinetic energy; that is, would be equal to the additional energy required to increase the existing velocity of every particle by one part in 2×10^{22}. When due account is taken of these results it is not surprising to find that in iron at a given temperature the specific resistance for a current-density of one ampere per cm.^2 differs from the specific resistance for an infinitesimal current-density by less than one part in 10^{12}.

The same remarks apply with even greater force to platinum and German silver, the other metals examined by Prof. Chrystal, since the *magnetic* influence of the current on the resistances of these metals must be far less than even in the case of iron.

From considerations similar to these, we should expect in all true conducting substances (even in those having marked magnetic properties) to find a sensibly linear relation connecting current-density with electromotive intensity in the neighbourhood of each point.

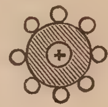
8. Contact E.M.F. and the Peltier Effect.

It will now appear that, by assuming in each molecule a mere arrangement of conducting and non-conducting parts, we may realize what is at all events a working model of contact E.M.F. and of the Peltier effect; and even should our model serve no other purpose, it directs our attention to a possibility which seems so far to have been overlooked, and which must, indeed, continue to be overlooked so long as each metal is regarded as homogeneous. As it not my object to state with becoming vagueness an hypothesis as to the nature of Peltier's phenomenon, but rather to picture as clearly as possible a mechanism whose principle may perhaps be suggestive of the truth, I shall assume for the molecules such a structure and distribution as appear most favourable to simplicity of treatment.

Suppose, then, that in one of the metals with which we

have to deal, each molecule is of the form indicated in fig. 2: a central perfectly conductive body, charged (say) positively, being completely surrounded by an insulating layer, and this again being partially (though not completely) enclosed by a number of perfectly conductive particles. It will be convenient to suppose that in each molecule these outer particles are all electrically connected with one another. When two such molecules come into collision, the outer particles of the one may not in general be reduced to the same potential as those of the other, for the electrical oscillations occasioned by the contact may not have time to subside into insignificance before the encounter is ended. But the general tendency of a collision between two molecules will be towards an equalization of the potentials of the outer particles, and the average values of potentials and charges amongst any considerable number of molecules will be the same as if, during each molecular encounter, this equalization has been completely effected.

Fig. 2.



It will here be convenient to introduce the term "conduction-potential," and as we proceed the following definitions will be found useful:—

The conduction-potential of a molecule is the potential (or average potential) of its outer conductive particles.

The conduction-potential at a point within a metallic body is the average conduction-potential of the molecules in the neighbourhood of that point.

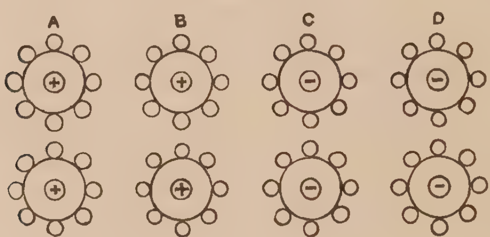
Let us now turn to the case of a considerable mass of metal, made up entirely of such molecules as that sketched in fig. 2; every molecule having the same structure and the same internal charge. Let us suppose also that the mass of metal is at the same temperature and in the same physical condition throughout, so that the average distance between adjacent molecules is the same in the neighbourhood of each point. If the metallic body is subjected to electrostatic induction, it is easy to see that no electric influence from without can penetrate far beyond the surface; for the outer particles of the superficial molecules, owing to their intermittent contacts with one another, will screen the interior of the metal as a network of continuous wires might do, and at a

depth of a very few molecules beneath the surface the screening will be practically complete. Thus any charge communicated to the conductor will be confined entirely to the superficial layers of molecules; and it follows that if between any two points well within the metal there were a difference of average conduction-potential, there would be a general flow of electrification from the higher towards the lower potential. The average conduction-potential within the metal does not therefore vary from point to point, and account being taken of the equality of the internal charges of the molecules, it follows that the average potential of the inter-molecular ether is similarly free from variation. This implies that the total electrification of any considerable assemblage of molecules within the metal is sensibly zero; so that the average charge on the outer conductive particles of each molecule is equal and opposite to the fixed charge on the central particle. The conclusions of this paragraph have been necessarily confined to molecules not too near the surface; for although the general tendency of an encounter between two molecules is always to equalize their conduction-potentials, these potentials change somewhat after the molecules have become separated, and near the free surface of the body the change will be systematically greater for the molecule which moves outwards after collision than for that which moves inwards.

We may now try to realize what will happen when contact takes place between two metals whose molecules are constructed on the general plan of fig. 2, while those of the one metal are not identical with those of the other. The most simple and intelligible view will be obtained by supposing the molecule of the one metal to have a positively charged central particle, while the molecule of the other has its central particle negatively charged (fig. 3). Consider what would occur if we could start from a condition in which each of the *border* molecules, A, B, C, D, &c., had *on the whole* no charge; a condition, that is, in which the outer particles of each molecule had a charge equal and opposite to the fixed charge of the central particle. (We have already seen that this is true for the *average* molecule *within* a homogeneous metallic mass.) Now when B and C come into collision, it is evident

that the outer particles of B will lose some of their negative electrification, while the outer particles of C will lose some of their positive; and thus we see that when the distribution of conduction-potential has become steady, such border particles as B or C will have on their outer particles a (negative or positive) charge *less* than the (positive or negative) fixed central charge, and to a smaller extent the same will be true

Fig. 3.



of particles (such as A, D, &c.) more remote from the border. But when the settled condition has been reached, the passage (say) of B backwards and forwards between A and C will not change the distribution of charges amongst the molecules; and the condition that no change of the kind shall take place is that when two molecules are in a position to collide their conduction-potentials shall be equal. Now when B approaches C (which has on the whole a negative charge) its potential is lowered, and when B returns towards A its potential rises again, *so that if* B *in its backward and forward motion is not to act as a systematic carrier of electrification between* A *and* C, *the conduction-potential of* A *must be higher than that of* C. Similarly, by considering C as moving backwards and forwards between B and D, we can see that B must have a higher conduction-potential than D.

This gives us *a contact-difference of conduction-potential.*

If we suppose our analysis to become a trifle less penetrating, such molecules as B (and to a less extent A, &c.) will appear to be positively electrified, C, D, &c. will appear to be negatively electrified, and molecules farther removed from the border will appear unelectrified. With still less microscopical vision, we shall find the conduction-potential constant

from point to point throughout the mass of metal on either side of the surface of separation; but as we approach that surface from the left, the conduction-potential begins to diminish, changing very rapidly by a finite amount as we pass through the boundary.

We now come to a theorem which is certainly true for our model, and which seems to me as certainly true for any mechanism which could be devised to represent the Peltier effect; but for the sake of avoiding questions of too controversial a character, the statement may be made in this conditional form:

Theorem IV.

In our model, the contact-difference of conduction-*potential between two metals is equal to the coefficient of the Peltier effect.* For when a molecule at the junction is moving backwards and forwards between places of different potentials, provided no current flows through the junction, as much electrification is carried from the lower to the higher potential as from the higher to the lower, and on the whole there is no transformation of electric energy into heat, or *vice versâ.* But when a current flows across the junction from the metal of lower to that of higher conduction-potential, the molecules at the junction are persistently carrying more electrification from the lower to the higher potential than they bring back with them on their return, and thus on the whole the movements of the molecules at the junction are systematically opposed by electrostatic forces. It is evident from elementary considerations that the quantity of electricity which has crossed the junction, multiplied by the step of (conduction-) potential up which it has passed, is the measure of the total work done by the molecules against electrostatic forces, and is therefore the measure of the heat absorbed. Similarly, when a current has been flowing from the metal of higher to that of lower conduction-potential, the quantity of electricity which has crossed the junction, multiplied by the (negative) step of conduction-potential, is the measure of the (negative) heat absorbed; that is, numerically, of the heat given out. Hence, in our model, the coefficient of the Peltier effect is equal to the contact-difference of conduction-potential.

Again, generally speaking, we may expect a difference of conduction-potential between a hotter and a colder portion of the same metal, owing to the increase of molecular distances which rise of temperature produces; and it is evident *that (on our model) the specific heat of electricity for any metal is equal to the rise of conduction-potential for one degree rise of temperature.*

9. From these results we pass on to

Theorem V.

For any pair of metals at the absolute zero of temperature, the Peltier effect vanishes.

This is evidently true for our model, for when the molecules are all reduced to relative rest, and there is permanent instead of intermittent contact amongst their outer particles, the conduction-potential will be uniform throughout both metals, and at the junction there will be no Peltier effect. But whatever view we take of the nature of the phenomenon, the proposition is necessarily true. For if the Peltier effect had a finite value for a pair of metals at the absolute zero of temperature, we could cause an absorption of heat by sending a current through the junction in the proper direction; and this is impossible, since there is no heat to be absorbed.

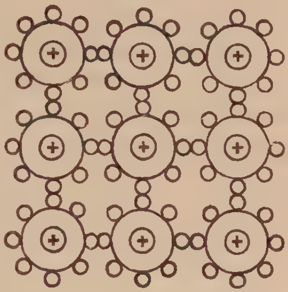
10. Volta E.M.F.'s.

We must now consider a possibility suggested by our model, and referred to in the opening sentence of §8. It is not difficult to see that, with molecules constructed on the plan of fig. 2, even when all measurements are made *in vacuo*, the conduction-potential of a mass of metal is not in general the same as the potential estimated by work done on an external charged body, or by electrification induced on a second mass of metal insulated from the first,—potential measured in the latter way being called for distinction the *induction-potential.*

We may realize this most easily by considering the case of two metals in contact at the absolute zero of temperature, for

then, in accordance with the last section, the Peltier effect at the junction vanishes, and the *conduction*-potential is the same throughout both metals; while on the other hand the difference of *induction*-potential may be finite. Let fig. 4 represent diagrammatically a very large number of molecules which are at rest with their outer conductive particles in electrical contact throughout. Let the fixed central charge of each molecule be positive. Then, if the outer conductive particles of each molecule formed a complete envelope around the central charge, the *induction*-potential of the metal would be identical with its *conduction*-potential, and the same as if the fixed central charges did not exist. But since we suppose the fixed charge in each molecule to be *incompletely* screened by the outer particles, it follows that at external points in the immediate neighbourhood of the metallic body the potential is raised above the conduction-potential by the fixed central charges. If these last were negative instead of positive, the potential just outside the metallic mass would be lower than the conduction-potential; and we may suppose that at any given temperature (such as the absolute zero with which we are dealing) the difference between the conduction-potential of a metallic body and the potential just outside the body depends upon the nature of the metal. Thus, even *in vacuo*, if two metals at the absolute zero of temperature be connected together so as to have the same conduction-potential, their induction-potentials may be different; and in general, whatever the temperature of the metals in contact, we may expect an inequality between difference of *conduction*-potential and the difference of *induction*-potential.

Fig. 4.



Before attempting to devise a model of Peltier's phenomenon and of electromotive forces of contact, I had held the opinion—in common, I believe, with the majority of disputants in the contact-force controversy—that the inductive measurement of potential-differences in a sufficiently perfect

vacuum must conclusively decide the points at issue. But if in reality there should be, as the model suggests, a difference between conduction-potentials and induction-potentials, we must not rely upon inductive experiments, even in a perfect vacuum, to determine the seats of electromotive force in a voltaic cell. For when we are dealing with the flow of currents through metals, it is the *conduction*-potential which concerns us.

11. The Transparency of Metals.

A difficulty in connexion with this subject is stated by Maxwell in the following well-known passage * :—" Gold, silver, and platinum are good conductors, and yet, when formed into very thin plates, they allow light to pass through them. From experiments which I have made on a piece of gold-leaf, the resistance of which was determined by Mr. Hockin, it appears that its transparency is very much greater than is consistent with our theory, unless we suppose that there is less loss of energy when the electromotive forces are reversed for every semi-vibration of light than when they act for sensible times, as in our ordinary experiments." Now we have seen that conduction is not a perfectly continuous phenomenon, but is due to innumerable encounters among perfectly conductive particles, and without entering upon any calculations (which indeed would be a difficult matter) we can see that there are, broadly speaking, two reasons why the opacity of metals is so much smaller than is indicated by Maxwell's analysis: these are, heterogeneity of structure and intermittence of contact.

To realize the influence of heterogeneity of structure without the complication of intermittent contacts, take the case of a metal at the absolute zero of temperature. We have then virtually to deal with a network of perfect conductors, constituting a body which as a whole has perfect conductivity, and of which even an excessively thin film would be an effectual barrier to electromagnetic waves, provided that the wave-

* 'Electricity and Magnetism,' 2nd ed. vol. ii. § 800. Wien (Wiedemann's *Annalen*, xxxv. pp. 41–62) found a silver film to have only such an opacity as would be deduced from about 1/440 of its actual conductivity.

length were great enough to justify us in treating the metal as homogeneous. But if we consider an extreme case, where the wave-length of the disturbance is negligible in comparison with the dimensions of a single conductive particle, a very thin layer of the metal will be far from absolutely opaque. For the conditions of the problem will then be the same as if we had ordinary luminous radiations obstructed by an agglomeration of perfectly reflecting bodies of appreciable size. Of course these extreme conditions are not realized in the case of the light transmitted by a metallic film; but if we may suppose that the diameter of a conductive particle is not quite negligible in comparison with a wave-length of light, it is clearly to be expected that very thin layers of the metal will fall short of that absolute opacity which in this case would follow from the assumption of homogeneity.

When we pass to the consideration of metals at ordinary temperatures, the conductivity for steady currents is finite; but for electromagnetic waves of short period we cannot even treat the metal as an agglomeration of finitely conductive particles continuously in contact with one another. It is evident that the shorter we make the period of the electromagnetic disturbance in comparison with the average inter-collisionary period of a (perfectly) conductive particle, the more nearly do the particles act as if permanently insulated from one another, and the less efficiently does the metal perform the functions of an electromagnetic screen.

Further considerations might be added concerning the average interchange of electrification between colliding particles when the electromotive intensity tending to produce such interchange is very rapidly alternating; but enough has been said to show that the opacity of conductors must be far less for luminous radiations than for electromagnetic disturbances of long period, and we may fairly expect, I think, that the transparency of metals is to be explained without attributing any new properties to the electromagnetic field.

The second part of this paper will deal with electrolytic conduction and disruptive discharge.

Note added April 30th.

In the course of the discussion Prof. S. P. Thompson objected to the arrangement of molecules in rectangular order, and he further suggested that the arguments might only be applicable in two dimensions. I had omitted to mention that the figures were intended to be sectional views of three-dimensional models, while the rectangular arrangement of molecules was merely adopted to save prolixity in the descriptions, and was so far from being essential to the investigation that the case of irregularly distributed coordinates and velocities was constantly before my mind. Another point raised by Prof. Thompson must also be considered here: in § 6 it does not necessarily follow that two conductive particles oppositely charged like A and B (fig. 1), approaching one another and subject to the influence of an external E.M.F. acting from right to left, would have the signs of their respective electrifications reversed by a *momentary* contact; in some encounters the readjustment of electrifications might even be in the opposite sense; but I think we may safely admit that *in the long run* the effect of innumerable collisions amongst such conducting particles as A and B will be to transfer electrification in the direction of the impressed E.M.F.

Prof. Rücker recalled a difficulty, which Lord Kelvin pointed out some time ago, in connexion with the collisions between molecules. If we suppose the molecules to be constituted like little pieces of elastic solid, every collision will cause some additional amount of translational energy to be converted into energy of vibration, and heat-energy will be continually running down into energy of shriller and shriller vibrations, that is, into energy of a lower form. In the foregoing pages, electrical contact between particles is supposed to occur during a collision, and Prof. Rücker remarked that the method suggested for avoiding an electromagnetic degradation of energy left untouched the corresponding mechanical difficulty. I have made some attempt to deal with this mechanical question in a previous paper*, where it was shown (§§ 10, 11) that, granted the fundamental assumption and an

* "A Theory concerning the Constitution of Matter," Proc. Phys. Soc. vol. xi. pp. 285, 286.

infinite propagation-velocity for gravitational stress, we may construct an atom having a finite number of freedoms. But in whatever way mechanical degradation of energy were eliminated, the difficulty of electromagnetic degradation would also have to be met, and without the doctrine laid down in Theorem I. there appeared to me to be no means of escape. Without making any assumption as to the constitution of a molecule or the nature of a collision, we may admit that in any body not absolutely cold there are particles in relative motion, so that two neighbouring particles are sometimes nearer together and sometimes farther apart. To realize the intermittence of contact required by Theorem II., we have only to suppose that when (but not until) the proximity of two particles has reached a certain limit electrification is capable of passing freely from one to the other.

The question of perfect or imperfect conductivity in the ultimate particles of bodies must be of importance in relation to the constitution of matter and its connexion with the ether; and whether or not the demonstrations above can be generally accepted as conclusive, the subject is certainly one which will repay further investigation.

Discussion.

Prof. S. P. Thompson thought the paper had an important bearing on the kinetic theory of solids. He saw no reason why Ohm's law should be proved, for he regarded it as a definition.

The President said the author represented all actions as being due to collisions, thereby introducing the same difficulties as were felt in the kinetic theory of gases, viz., that collisions would give rise to mechanical oscillations in the molecules of shriller and shriller pitch. Prof. J. J. Thomson had recently given an explanation of electrical phenomena by vortex filaments.

After some remarks on the visibility of molecules by Mr. Hovenden,

Dr. Burton, in reply to Prof. Thompson, said Ohm's law, when expressed as

$$\frac{\mathrm{E}}{\mathrm{C}} = a \textit{ constant},$$

was really a law, and not a mere definition.

XIII. *The Minimum Temperature of Visibility.* *By* P. L. Gray, *B.Sc., Assoc. R.C.S., Lecturer in Physics, Mason College, Birmingham**.

The title of this paper is, for convenience, somewhat elliptical. Fully, it should be "The Minimum Temperature at which a Strip of Platinum, either bright or lamp-blacked, becomes visible in the dark."

Apparently no exact work has been done on the point since that of Draper ("On the Production of Light by Heat," Phil. Mag. 1847, xxx. p. 345), who used a somewhat similar method to that described in this paper. This work seems to be generally little known, vague statements, usually without authority, being given when the subject is referred to at all. It only fell under the present writer's notice after the observations, detailed below, had been completed.

Draper looked at a small strip of platinum, only $\frac{1}{20}$ inch wide, in a dark room, gradually heating the metal by an electric current until it became visible, and estimating its temperature from its linear expansion, measured by a magnifying-lever method. Taking a mean coefficient of expansion, from Dulong and Petit's results, with a lengthening of $\frac{1}{222}$, he arrives at a temperature of 525° C. This is obviously too high, as no account is taken of the increase of the coefficient with temperature. Taking the most recent result† for platinum,

$$\alpha = 10^{-9}(8901 + 1{\cdot}21\,t),$$

or

$$\frac{dl}{dt} = 10^{-9}(8901 + 1{\cdot}21\,t);$$

and integrating, we have

$$l_t - l_0 = 10^{-9} \times 8901\,t + 10^{-9} \times 1{\cdot}21\,\frac{t^2}{2},$$

or

$$\frac{1}{222} = {\cdot}0045045 = \quad " \quad " \quad "$$

whence $t = 490°$ C.

* Read April 13, 1894.

† J. R. Benoit, *Travaux et Mémoires du Bureau International des Poids et Mesures*, vi. 1888, p. 90.

As this formula was only tested experimentally up to about 73° C., it is most likely that another term, in t^2, should be added, which would somewhat reduce this value, and thus produce a satisfactory agreement between Draper's result and the maximum obtained by myself.

In the same paper Draper arrives at the conclusion that *all solid bodies become visible at the same temperature*; a conclusion which is fully borne out by the observations given below, as to the equality of temperatures in the case of bare and lamp-blacked platinum.

Draper's value being, as shown above, somewhat doubtful, the point seems worthy of more exact experiment, both from a physical and a physiological point of view; and as I had, in beginning a series of investigations on Radiation, a means of keeping a platinum surface at any desired temperature, I made a number of observations on the subject, while waiting for other apparatus for use with which the platinum strip was primarily intended.

The instrument in question is Wilson and Gray's modified form of Joly's Meldometer, and is described in their paper, "Experimental Investigations on the Effective Temperature of the Sun," read before the Royal Society on March 15th, and shortly to be published*. It consists essentially of a strip of very thin platinum, about 10 centim. long, 1 centim. broad, and $\frac{1}{50}$ millim. thick. The plane of the strip is vertical.

It can be heated by an electric current, and its linear expansion is indicated by an optical method, by which an alteration in temperature of 1° can easily be noticed. The method of calibration is described in Joly's paper †, and in that already mentioned, so that it is unnecessary to do more than briefly refer to it here. Minute fragments of substances of known melting-points are placed on the strip and watched through a microscope, while the temperature is very slowly and cautiously raised until, in any case, melting is seen to take place, when the position of the spot of light which indicates the expansion of the strip is noted. In these experiments the substances used were K_2NO_3 (339°), AgCl (451°), KBr

* Phil. Trans. 1894.

† Proc. R. I. A. vol. ii. 3rd series, 1891–92, p. 38.

(699°), and gold (1041°) *. From these observations a curve showing the relation between temperature and scale-readings is obtained. One point may, however, be mentioned, viz. as to the difference of temperature between the middle of the strip and its surface. Joly says † that the difference is probably measured by hundredths of a degree only. The following figures will show the correctness of this assumption.

At a temperature of about 500° C., amount of energy lost per second per square centimetre $= \cdot 3$ calorie.

Thickness of section $= \frac{1}{50}$ millim. $= \cdot 002$ centim.

∴ Half " " $=$ $\cdot 001$ "

Flow of heat $= \frac{k\theta}{d}$, where $k =$ conductivity $(= \cdot 2)$, and $\theta =$ the difference of temperature required, at the ends of the distance d $(= \cdot 001$ centim.),

$$\therefore\ \cdot 3 = \frac{\cdot 2\theta}{\cdot 001}, \quad \text{whence } \theta = 0^\circ \cdot 0015.$$

The loss of energy per unit area was obtained by a voltmeter-and-ammeter method, which is to be further developed for the Radiation experiments. All the figures are approximate, the smallness of the resulting difference (about $\frac{1}{1000}$ ° C.) showing that there is here no need for any high degree of accuracy.

With the surface of the platinum lampblacked the case is different; but even here an approximate calculation shows that there is still less than 1° difference between the platinum and the surface of the carbon. In a typical case the following results were found:—

Thickness of layer $= \cdot 00033$ centim. This was determined from a knowledge of the specific gravity of the lampblack.

Conductivity $= \cdot 0002$. This is the value given by Everett for powdered carbon, and is probably *less* than the correct value for the lampblack; but the two must be of about the same order of magnitude.

Loss of heat per unit area at 500° C. $= \cdot 5$ calorie per second;

$$\therefore\ \cdot 5 = \frac{\cdot 0002\,\theta}{\cdot 00033}; \quad \therefore\ \theta = 0^\circ \cdot 8.$$

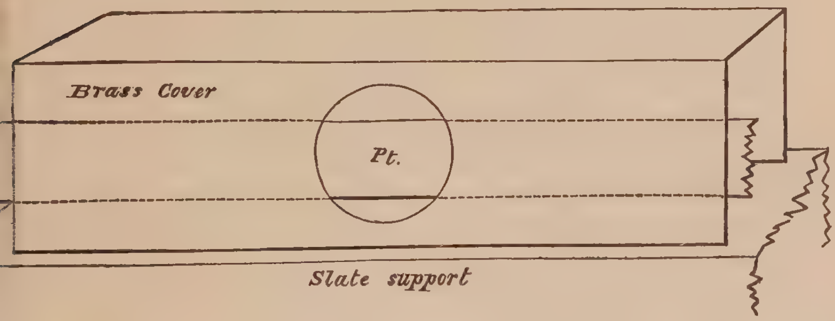
* See the same two papers already quoted.

† Proc. R. I. A. vol. ii. 3rd series, 1891–92, p. 49.

Method of Making the Experiments.

The first requisite was to get the strip in a perfectly dark enclosure, within which both eyes could be directed towards it without strain. To this end the apparatus was enclosed in a wooden box (blackened within), one end of which was replaced by a black velvet cloth, under which the observer placed his head, and which he could gather round his neck and under his chin so that not a ray of light could penetrate the enclosure. The box was about 48 centim. long, 30 broad, and 22 high, and ordinarily the eyes, in making an observation, would be about 30 centim. from the strip. The other end of the box was provided with a hinged shutter, which was lifted immediately after an observation had been made, for the purpose of noting the temperature of the strip.

The strip itself was further protected from draughts &c. by means of a piece of brass, bent twice at right angles, and resting on the slate block below the strip, as in the calibration experiments; thus (natural size) :—



The angular dimensions of the surface of platinum, as seen in any experiment, were therefore :—

$$\begin{aligned}\text{Apparent length} &= 3^\circ\ 49' \text{ approximately;}\\ \text{,, \quad width} &= 1^\circ\ 54' \quad\text{,,}\end{aligned}$$

so that the apparent area subtended was about 36 times that of the full moon.

The current by which the strip was heated ran through a variable carbon resistance, the handle of which was within

convenient reach of the observer as he sat with his head under the black cloth. He could thus alter the temperature of the platinum until it was on the very verge of invisibility, a very small fraction of a turn being then sufficient to produce utter darkness where before the area of faint light had been. A contact-breaker was also within convenient reach, so that the current could be broken or made at pleasure, and the objective reality of the faint luminosity at the limiting-point thus demonstrated. When he was satisfied that the limiting-point had been reached the hinged end of the box was opened, a beam of light sent to the mirror connected with the strip, and the deflexion, giving the temperature, read on the scale. The possible error in the estimation of the absolute value of the temperature may be taken as certainly not more than 2°.

Discussion of the Results.

My first idea, in putting down final results, was to take the mean of a large number of observations as expressing the required minimum temperature; but there soon appeared to be too much variation in individual cases, both for the same eyes at different times and for different people's eyes, for this mean to be of any value. It seems better, therefore, to put down the determinations in some detail, with such remarks as may help towards some general conclusions afterwards.

The first case given is my own, in which it has naturally been easier to obtain a larger number of records than in the case of other people.

The dates are given merely to distinguish one day's observations from another's.

The letter B after any temperature signifies that the surface was lampblacked in that experiment; otherwise it was bare, and in its ordinary condition of polish.

The details of the observations are as follows:—

Feb. 10th. 451°, 460°, 460°, 444°, 447°, 448°. Observations made in the morning, without any special preparation as to resting the eyes.

„ 14th. 427°, 422°, 422°. Evening.

„ 15th. 437° (B), 432° (B), 427° (B), 427°, 427°. Evening.

Feb. 16th. 422°, 418°, 421°, 419° (B), 419° (B). Evening; 2nd observation after two minutes in complete darkness.

„ 19th. 410°, 410°. Evening.

„ 22nd. 409°, 409°, 408°. Evening.

After these observations more attention was paid to the time of day and the state of preparation of the eyes.

Feb. 26th. 401°, 397°; 388°, 394°, 384°; 398°, 403°. Evening. Between 2nd and 3rd observations, 11 minutes were passed in perfect darkness; after the 5th, 10 minutes in writing under a bright incandescent lamp.

Before the next set, 16 minutes with eyes shut in nearly dark room :—

383°, 392°, 392° (B), 409° (B), 409° (B), 416°, 417°, 409°.

„ 27th. 470°, 464°, 461°, 463°. Morning (bright).

413°, 414°; 392°, 400°, 400°; 410°. Evening; 3rd observation after half an hour with eyes shut. Several more observations were made, showing a regular fall of temperature after resting the eyes in the dark, and a rise after reading &c., the limits being about 390° and 410°.

„ 28th. 455°, 447°, 447°; 459°. Morning (dull).

„ „ 453°; 417°, 415°, 419°. Afternoon; 4 minutes in darkness after 1st observation.

„ „ 407°; 389°, 389°; 404°, 399°; 392, 392°, 393°. 16 minutes in darkness between 1st and 2nd observations; 12 minutes' reading between 3rd and 4th, and again a few minutes' rest after the 5th.

Observations on two succeeding days gave readings varying from 470° in the morning to 385° in the evening, the influence of rest in darkness being always plainly marked.

To test if the intervals of rest noted above were sufficient to bring the eyes to their extreme state of sensitiveness, some observations were made at 3 A.M., after I had been asleep in a dark room for 3 hours. The readings then obtained were 373°, 386°, showing a slightly greater sensitiveness than that

in any previous experiment; the small difference probably indicates that the eyes were very near their extreme limit, and that no longer rest would give a lower reading.

Before going on to the general conclusions to be drawn from these results, the values are given obtained by other observers, who (with the exception of two) were either members of the staff, or students, of Mason College. They are probably not equally trustworthy, and in general no particular preparation was gone through by the observer.

Usually the first observation in each case showed a higher temperature than the second and succeeding ones. This was to be expected, since the time spent in darkness during the first experiment prepares the eye to a certain extent for the second. The results obtained are as follows :—

Case A.	Mean temperature . .	432° C.
„ B.	„ „ . .	422
„ C.	„ „ . .	436
„ D.	„ „ . .	409
„ E.	„ „ . .	428
„ F.	„ „ . .	438
„ I.	„ „ . .	426
„ J.	„ „ . .	440

All the above were taken either in the morning or early in the afternoon.

Case H. Mean temperature . . 408°.

In this case the room was darkish.

Case G.	Evening	388°
	Morning	435°
	Evening	380°–385°

The last observation was made after 17 minutes had been spent by the observer with his eyes shut in the nearly dark laboratory.

General Conclusions.

(1) *That the minimum temperature of visibility is the same for a bright polished metallic surface as for one covered with lampblack,* although the intensity of the radiation in the two cases may be different.

This result may at first be, to some, unexpected, but a little consideration will show that it might have been, *à priori*, anticipated. For probably temperature governs the highest wave-length from a radiating body, and wave-length governs visibility, at least after an extremely small intensity of radiation has been passed *.

(2) *That the visible limit at the red end of the spectrum varies greatly for a normal eye, according to its state of preparation*; *i.e.* according to the intensity of the light in which the observer has been before making the observation.

I take my own eyes as normal; they have been tested in the Anthropometrical Laboratory at S. Kensington, and roughly by Captain Abney's method; and in the figures given above the results are supported by those obtained by the other "cases," none of whom † are known to have abnormal sight.

Speaking generally, we may say that a bright light diminishes the sensitiveness of the eye to radiation of low frequency; that darkness increases it. Or that, as a rule, the eye is less sensitive in the morning than at night.

(3) *That for the less sensitive condition, the minimum temperature of visibility for the surface of a solid is about* 470° C., *but that this may be much reduced by even a few minutes in a dark room.*

(4) *That at night, a surface at a temperature of* 410° *is visible, and that by resting the eyes in complete darkness, this may be reduced to as low as* 370° *nearly, below which apparently one cannot go*, since 10 minutes' rest appears to be almost as efficacious as 3 hours'.

(5) *That different people's eyes (of no special or known departure from normality) differ somewhat in their "minimum temperature of visibility," but probably not to any great extent, if tested under the same conditions as to preparation, &c.*

Case G was a somewhat curious one; the observer was the

* Langley, "Energy and Vision," Phil. Mag. xxvii. (1889), shows that the amount of energy sufficient to excite vision is immensely less than that radiating from the strip in these experiments, at least in the low-red wave-lengths.

† With the possible exception of Case G.

one exception among those I tried whose red colour-perception is perhaps not quite normal. He arrived at the lowest temperature in the evening without any long rest in the darkness, and, as is seen above, 17 minutes in perfect darkness produced no alteration; yet, in the morning, he went up to about the usual figure, so that what I had hoped to find an abnormal case turned out approximately ordinary.

The loss of distinct *colour* at the low temperatures is very striking; the appearance to myself, and to most of the observers, has absolutely nothing of red in it, but is like a white mist—the nearest comparison I can make.

In the *morning* observations, however, when the strip disappeared at from 460° to 470°, the last appearance was distinctly reddish; and this agrees with one observation noted at night, when after getting the visibility critical-point at about 390° C., the temperature was raised until one could declare for certain that the light looked red: it was then found to be 449°.

Of course, in all the observations, the luminous area was most distinctly seen by somewhat averting the gaze from it; generally I found it best to look in the direction of either far upper corner of the enclosure.

As already mentioned, most of the observers pronounced the appearance at the critical-point to be that of a "whitish mist;" one, however, thought he saw a slight "lilac tinge" in it; and "Case G" declared it to be decidedly yellow, which is interesting, because to him a red mark on white paper (such as a pip on a card belonging to one of the red suits of a pack) appears yellow, by artificial light at night.

In one experiment a plate of glass, $\frac{1}{8}$ inch thick, and in another a layer of water, $\frac{1}{2}$ inch thick, were inserted between the strip and the eye, without making the slightest difference in the phenomenon; showing (1) that the point where these substances begin to be more or less opaque to infra-red radiation had not been reached; (2) that the small difference in

intensity produced by their insertion had no appreciable effect. This last conclusion is far more strongly borne out by the equality of temperature in the case of the bare metallic and the black surfaces, and indicates that in all the cases it was *wave-length*, and not *intensity*, which was determinative of visibility, so disposing of the possible objection that the difference between "morning" and "evening" might be due merely to the state of enlargement of the pupil of the eye, which would naturally be more contracted at the one time than at the other, thus affecting the total amount of radiation falling on the retina. Also, if such an objection were valid, it would imply that fatigue of the muscles of the iris produced a relatively enormous "time-lag" in following changes of luminous intensity, which we know does not exist.

There seems, in fact, to be little doubt that the difference is due to the retina itself becoming sensitive to long waves after rest, which were incapable of affecting it when it was in some way fatigued by exposure to the ordinary bright light of day.

The next and obvious step is to find the respective wave-lengths corresponding to the different temperatures. This point, however, and others, cannot be determined without some additions to the present apparatus, and will form the subject of a future paper.

Discussion.

Mr. Blakesley inquired if the author had tried condensing the light from the strip. As to colourlessness, he observed that the parts of the retina active in oblique vision were less sensitive to colour than the central portions.

Dr. Burton remarked that in the experiments the presence of light and not colour was being observed. When illumination was faint, as in twilight or moonlight, it was very difficult to distinguish colours. In the solar spectrum one did not see any whitish termination at the red end.

Mr. Elder said that Captain Abney had shown that all colours appear grey when of small intensity.

The President thought the question as to whether visibility depends on wave-length or on energy was an important one; probably a minimum amount of energy was essential. At such low temperatures the emission curves of the different wave-lengths may not have become sufficiently separated to be distinguished.

Mr. Gray, in reply, said Prof. Langley had shown that a minimum, but very small, amount of energy was necessary to vision in all parts of the spectrum.

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